METRICS AND APPROXIMABILITY FOR TRIANGULATED CATEGORIES

Introduction

Triangulated categories are ubiquitous in algebra, topology, and geometry. Neeman has recently introduced certain metric/analytic techniques for triangulated categories in a series of articles [Nee21b, Nee18, Nee26, BNP23, Nee24a, CHNS24, Nee25][1], and used them to settle multiple conjectures concerning triangulated categories arising in algebraic geometry, and to prove other new results[2].

The idea of this mini-course is to give a feeling of these techniques in action, by going through some of the interesting results proven using them. Most of the applications so far are in algebraic geometry, although these techniques are slowly being applied in representation theory and homotopy theory also, see for example [CG24, Mat26a, Mat26b, LS25].

In particular, we will look at the connection between the regularity of a Noetherian scheme with the existence of bounded t-structures and strong generators for the category of perfect complexes. Further, we will discuss some new representability theorems and how to use them to give a unified proof of GAGA theorems in algebraic geometry.

In the second half, we will prove that the derived category $D_{\mathrm{Qcoh}}(X)$ of a Noetherian separated scheme is approximable. We will then move on to the passage between various triangulated subcategories of a weakly approximable triangulated category, which is a vast generalisation of a result by Rickard on derived Morita theory. Finally, we will end with a result on the bijection of semiorthogonal decompositions on various (small) triangulated categories associated to a scheme.

Prerequisites. We will assume familiarity with the basics of triangulated categories, or their enhanced incarnations (stable model/$(\infty,1)$-categories, pretriangulated DG-categories ...). As most of the applications we will discuss are algebro-geometric, we will expect some familiarity with the language of schemes, and with the derived categories and functors associated to them.

Conventions. We will use the following conventions and notations without mention in the rest of the document.

1. All rings will be commutative unless explicitly mentioned otherwise.
2. Our gradings will be cohomological. For full subcategories $\mathcal{A}_1, \mathcal{A}_2$ of a triangulated category $\mathcal{T}$, we denote by $\mathcal{A}_1 * \mathcal{A}_2$ to be the full subcategory consisting of objects A for which there exists a triangle $A_1 \to A \to A_2 \to A_1[1]$ with $A_i \in \mathcal{A}_i$.
3. When working with triangulated categories associated to a scheme, we will reserve the notation $\tau^{\leq 0}, \tau^{\geq 0}$ and $\sigma^{\leq 0}, \sigma^{\geq 0}$ for the so called canonical and stupid truncations respectively [Sta24, Tag 0118].
4. We will "exact triangles" and "triangles" interchangeably.

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[1] In the order the preprints appeared on arXiv.

[2] See also the survey articles [Nee20, Nee21a, Nee23, CNS25].

1. BOUNDED T-STRUCTURES AND REGULARITY

Originally introduced in the early 80's by Beilinson, Bernstein, and Deligne to study perverse sheaves, t-structures have by now become important tools to study triangulated categories. Of particular importance are bounded t-structures, which play a central role in the theory of Bridgeland stability conditions. Further, in the theory of motives, the conjectured motivic (bounded) t-structure is of central importance. For completeness, we begin by the definition, followed by a prototypical example.

Definition 1.1. Let $\mathcal{T}$ be a triangulated category. A *t-structure* on $\mathcal{T}$ is a pair of strictly full subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$[3] such that

1. $\mathcal{T}^{\leq 0} \subseteq \mathcal{T}^{\leq 0}[-1]$ and $\mathcal{T}^{\geq 0} \subseteq \mathcal{T}^{\geq 0}[1]$,
2. $\operatorname{Hom}(A, B) = 0$ for all $A \in \mathcal{T}^{\leq 0}$ and $B \in \mathcal{T}^{\geq 0}[-1]$,
3. for all $T \in \mathcal{T}$, there exists a triangle $A \to T \to B \to A[1]$ with $A \in \mathcal{T}^{\leq 0}$ and $B \in \mathcal{T}^{\geq 0}[-1]$.

For notational convenience, we define $\mathcal{T}^{\leq n} := \mathcal{T}^{\leq 0}[-n]$ and $\mathcal{T}^{\geq n} := \mathcal{T}^{\geq 0}[-n]$. Finally, we say that a t-structure is *bounded* if for all $T \in \mathcal{T}$, there exists $n(T) \geq 0$ with $T \in \mathcal{T}^{\geq -n(T)} \cap \mathcal{T}^{\leq n(T)}$.

Remark 1.2. It turns out that the triangles we get in Theorem 1.1 are functorial. In fact, we get a functor $\tau^{\leq 0} \colon \mathcal{T} \to \mathcal{T}^{\leq 0}$ (resp. $\tau^{\geq 0} \colon \mathcal{T} \to \mathcal{T}^{\geq 0}$) which is the right (resp. left) adjoint to the inclusion. Furthermore, we get a triangle $\tau^{\leq 0}T \to T \to \tau^{\geq 0}T \to \tau^{\leq 0}T[1]$ for any $T \in \mathcal{T}$ with the first (resp. second) map given by the counit (resp. unit) of adjunction. This triangle, called the *truncation triangle for T*, is canonically isomorphic to the triangle in Theorem 1.1(3).

It turns out that the intersection $\mathcal{T}^{\heartsuit} := \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$, called the *heart of the t-structure*, is an abelian category. Furthermore, the functor

$$\tau^{\leq 0} \circ \tau^{\geq 0} \simeq \tau^{\geq 0} \circ \tau^{\leq 0} \colon \mathcal{T} \to \mathcal{T}^{\heartsuit}$$

is a homological functor, that is, it sends triangles to long exact sequences.

Example 1.3. Let $\mathcal{A}$ be an abelian category and $\mathcal{B} \subseteq \mathcal{A}$ any Serre subcategory[4]. Then, $D_{\mathcal{B}}(\mathcal{A})$, which is the derived category of $\mathcal{A}$ with cohomology in $\mathcal{B}$, has a t-structure[5] $(D_{\mathcal{B}}(\mathcal{A})^{\leq 0}, D_{\mathcal{B}}(\mathcal{A})^{\geq 0})$ defined by

$$D_{\mathcal{B}}(\mathcal{A})^{\leq 0} = \{F : H^i(F) = 0 \text{ for all } i \geq 1\},\ D_{\mathcal{B}}(\mathcal{A})^{\geq 0} = \{F : H^i(F) = 0 \text{ for all } i \leq -1\},$$

with truncation triangles given by the so called canonical truncation [Sta24, Tag 0118]. This restricts to a bounded t-structure on $D^b_{\mathcal{B}}(\mathcal{A}) = \{F : H^i(F) = 0 \text{ for } |i| \gg 0\}$.

In particular, let X be a Noetherian scheme. Let $\mathcal{A} = \operatorname{Mod}(\mathcal{O}_X)$ be the (Grothendieck) abelian category of $\mathcal{O}_X$-modules. Then,

1. taking $\mathcal{B} = \operatorname{Qcoh}(X)$ the Serre subcategory of quasi-coherent sheaves, we get a t-structure $(D_{\operatorname{Qcoh}}(X)^{\leq 0}, D_{\operatorname{Qcoh}}(X)^{\geq 0})$ defined by

$$D_{\operatorname{Qcoh}}(X)^{\leq 0} = \{F : \mathcal{H}^i(F) = 0 \text{ for all } i \geq 1\},\ D_{\operatorname{Qcoh}}(X)^{\geq 0} = \{F : \mathcal{H}^i(F) = 0 \text{ for all } i \leq -1\}.$$

2. taking $\mathcal{B} = \operatorname{coh}(X)$ the Serre subcategory of coherent sheaves, this t-structure restricts to a bounded t-structure on $D^b_{\operatorname{coh}}(X)$, the derived category of sheaves with bounded and coherent cohomology.

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[3] Although sometimes we will denote a t-structure by $(\mathcal{U}, \mathcal{V})$.

[4] That is, an abelian subcategory such that for any exact $M_1 \to M \to M_2$, $M_1, M_2 \in \mathcal{B} \implies M \in \mathcal{B}$.

[5] This is well known, see for example [GM03, IV.4, Proposition 3].

We begin our story with a beautiful result by Antieau, Gepner, and Heller which provides an obstruction to the existence of bounded t-structures.

Theorem 1.4 ([AGH19, Theorem 1.1])**.** *If $\mathcal{T}$ is an (enhanced) essentially small triangulated category with a bounded t-structure then the first negative K-theory of $\mathcal{T}$ vanishes.*

Based on this result, the authors proposed the following conjecture.

Conjecture 1.5 ([AGH19, Conjecture 1.5])**.** Let X be a singular Noetherian finite-dimensional scheme. Then, $\operatorname{Perf}(X)$ has no bounded t-structures.

Let us review some of the evidence for (and against) the conjecture from what is known about the K-theory of schemes[6].

Discussion 1.6. Let X be a Noetherian scheme. Then,

1. if X is regular then $K_{-n}(X) = 0$ for all $n \geq 1$ by the work of Schlichting [Sch06, Example 9.6],
2. if X is singular, the negative K-groups are often non-trivial,
3. but, if X is 0-dimensional, then $K_{-n}(X) = 0$ for $n \geq 1$.

The above might not instill too much belief for the conjecture, but thanks to Neeman, we now know that it is in fact true! Moreover, Neeman proves a relative version; to state it recall that for a closed subset Z of a Noetherian scheme X, $\operatorname{Perf}_Z(X)$ denotes the category of perfect complexes supported on Z.

Theorem 1.7 ([Nee24a, Theorem 0.1])**.** *Let X be a Noetherian finite-dimensional scheme and $Z \subseteq X$ a closed subset. Then, $\operatorname{Perf}_Z(X)$ has a bounded t-structure if and only if $Z \subseteq \operatorname{reg}(X)$.*

The proof of this result will occupy the rest of this section. Let us begin with the following categorical characterisation of regularity[7].

Lemma 1.8. *Let X be a Noetherian scheme and Z a closed subscheme. Then, $Z \subseteq \operatorname{reg}(X)$ if and only if the natural inclusion $\operatorname{Perf}_Z(X) \to D^b_{\mathrm{coh},Z}(X)$ is an equivalence.*

Proof. Assume that $Z \subseteq \operatorname{reg}(X)$. Let $F \in D^b_{\mathrm{coh},Z}(X)$. By our assumption, we know that either F_x is zero or a bounded complex with finitely generated cohomology over a regular local ring and hence perfect[8]. As F is bounded, this implies that F is in fact a perfect complex[9].

Conversely, assume that the natural inclusion $\operatorname{Perf}_Z(X) \to D^b_{\mathrm{coh},Z}(X)$ is an equivalence. Let $x \in Z$ be any closed point. Consider the skyscraper sheaf $k(x)$ at x, which lies in $D^b_{\mathrm{coh},Z}(X)$. Hence, by our hypothesis, it lies in $\operatorname{Perf}_Z(X)$, which in turn implies that $k(x)$ has finite projective dimension as an $\mathcal{O}_{X,x}$-module. And so, $\operatorname{gl.dim.}(\mathcal{O}_{X,x}) = \operatorname{proj.dim.}(k(x)) < \infty$; that is, $\mathcal{O}_{X,x}$ is a regular local ring. As regularity is closed under generalisation, we get that $Z \subseteq \operatorname{reg}(X)$. $\square$

Remark 1.9. In light of the previous result, we need to show that $\operatorname{Perf}_Z(X)$ has a bounded t-structure if and only if the natural inclusion $\operatorname{Perf}_Z(X) \to D^b_{\mathrm{coh},Z}(X)$ is an equivalence. As

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[6] For a detailed discussion, see the relevant portions of [Nee24b].

[7] Which is well-known to those who know it well.

[8] As can be seen by taking a projective resolution P of F_x by finitely generated projectives and considering $\sigma^{\leq n}P$ where n is the minimal non-zero cohomology of P.

[9] One way to show this is go affine local, and use an argument as the footnote above using the fact that for any module M and $n \geq 0$, the subset $\{x : \operatorname{proj.dim.}(M_x) \leq n\}$ is open, see [IT19, Lemma 2.3].

$D^b_{\mathrm{coh},Z}(X)$ always has a bounded t-structure, one of the implications is trivially satisfied. What remains to show is that $\mathrm{Perf}_Z(X)$ having a bounded t-structure implies that the natural inclusion $\mathrm{Perf}_Z(X) \to D^b_{\mathrm{coh},Z}(X)$ is an equivalence.

A tale of two t-structures.

Notation 1.10. Suppose $\mathcal{A}$ is a subset of $\mathrm{Perf}_Z(X)$, and let $\mathcal{B} = \bigcup_{n \geq 0} \mathcal{A}[n]$. Then, we can define a t-structure $(\mathcal{T}^{\leq 0}_{\mathcal{A}}, \mathcal{T}^{\geq 0}_{\mathcal{A}})$ with $\mathcal{T}^{\leq 0}_{\mathcal{A}} := \mathrm{Coprod}(\mathcal{B})$ and $\mathcal{T}^{\geq 0}_{\mathcal{A}} := \mathcal{B}^{\perp}[1]$ on $D_{\mathrm{Qcoh},Z}(X)$ by [ATJLS03, Theorem A.1]. Here $\mathrm{Coprod}(\mathcal{B})$ denotes the subcategory of $\mathcal{T}$ generated by $\mathcal{B}$ which is closed under coproducts and extensions. Further, for any object $E \in \mathcal{T}$, we define $(\mathcal{T}^{\leq 0}_{E}, \mathcal{T}^{\geq 0}_{E}) := (\mathcal{T}^{\leq 0}_{\{E\}}, \mathcal{T}^{\geq 0}_{\{E\}})$.

Suppose $\mathrm{Perf}_Z(X)$ has a bounded t-structure $(\mathcal{U}, \mathcal{V})$. Then in the notation above, we can construct a t-structure $(\mathcal{T}^{\leq 0}_{\mathcal{U}}, \mathcal{T}^{\geq 0}_{\mathcal{U}})$ on $D_{\mathrm{Qcoh},Z}(X)$ which restricts back to the t-structure $(\mathcal{U}, \mathcal{V})$ on $\mathrm{Perf}_Z(X)$. Note that $D_{\mathrm{Qcoh},Z}(X)$ already has the so called "standard t-structure" defined via canonical truncations which we denote by $(D_{\mathrm{Qcoh},Z}(X)^{\leq 0}, D_{\mathrm{Qcoh},Z}(X)^{\geq 0})$. The main technical ingredient in proving our main result will be a comparison between these two t-structures. To show why this is enough, we begin by considering the toy case in which the two t-structures agree, or equivalently, when the standard t-structure restricts to the category of perfect complexes[10].

Proposition 1.11. *Let X be a Noetherian scheme and $Z \subseteq X$ a closed subset such that the standard t-structure on $D_{\mathrm{Qcoh},Z}(X)$ restricts to $\mathrm{Perf}_Z(X)$. Then, $Z \subseteq \mathrm{reg}(X)$, or equivalently, the natural inclusion $\mathrm{Perf}_Z(X) \to D^b_{\mathrm{coh},Z}(X)$ is an equivalence.*

We will look at two proofs of this. The first is perhaps a bit easier, but the second more amenable to a generalisation which we will need later.

Proof 1 of Theorem 1.11. Let $x \in Z$ be a closed point, and $j\colon U = \mathrm{Spec}(R) \to X$ be any affine open neighbourhood of x. Let $K(x)$ be the Koszul complex on x[11], and consider $\widetilde{K(x)} := Rj_*K(x)$. By staring intently at the equivalence $Lj^*\colon D_{\mathrm{Qcoh},\{\mathrm{x}\}}(X) \leftrightarrows D_{\mathrm{Qcoh},\{\mathrm{x}\}}(U) : Rj_*$[12], we see that $\widetilde{K(x)} \in \mathrm{Perf}_Z(X)$ and $\tau^{\geq 0}\widetilde{K(x)} = k(x)$ where $k(x)$ denotes the skyscraper sheaf at x. So, by our hypothesis $k(x)$ is a perfect complex, and hence has finite projective dimension as a $\mathcal{O}_{X,x}$-module. And so, $\mathrm{gl.dim.}(\mathcal{O}_{X,x}) = \mathrm{proj.dim.}(k(x)) < \infty$; that is, $\mathcal{O}_{X,x}$ is a regular local ring. As regularity is closed under generalisation, we get that $Z \subseteq \mathrm{reg}(X)$. □

Before we get to the second proof, we need the following result which was originally proven for $Z = X$ in [LN07].

Lemma 1.12 ([Sta24, Tag 08EL])**.** *Let X be a Noetherian scheme with a closed subset Z. Then, for any $F \in D^-_{\mathrm{coh},Z}(X)$ and any integer n, there exists $E \in \mathrm{Perf}_Z(X)$ and a triangle $E \to F \to D \to E[1]$ with $D \in D_{\mathrm{Qcoh}}(X)^{\leq -n}$.*

This is in general a difficult and somewhat technical result. We give a proof for the special case $Z = X$ when there are enough vector bundles (that is, every coherent sheaf is the quotient of a locally free sheaf of finite rank) as an illustration.

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[10] See [BL26, Proposition 2.5] for a related result for ring spectra.

[11] That is, choose some finite set of generators $\{f_1, \cdots, f_n\}$ for the ideal defined by x, and set $K(x) := K(f_1, \cdots .f_r) = \bigotimes_{i=1}^r \mathrm{Cone}(R \xrightarrow{f_i} R)$.

[12] See [Nee24a, Corollary 6.4] for a proof of this equivalence, though intuitively this should be clear as $\{x\} \subseteq U$.

Proof of Theorem 1.12 for $Z = X$ when there are enough vector bundles. What are hypothesis means is that for every coherent sheaf, there is a vector bundle surjecting onto it. This almost immediately implies that for any object in $D^-_{\mathrm{coh}}(X)$ is quasi-isomorphic to a complex P such that $P^i = 0$ for $i > m$ and each P^i is a vector bundle. We then get the required triangles by the so called "brutal" or "hard" truncations of P. ☐

Proof 2 of Theorem 1.11. It is enough to show that every object in $D^b_{\mathrm{coh},Z}(X)$ is a perfect complex. Let $F \in D^b_{\mathrm{coh},Z}(X)$. By replacing F by $F[n]$ for some $n \ll 0$ if necessary, we may assume that $F \in D^b_{\mathrm{coh},Z}(X)^{\geq 0}$. By Theorem 1.12, there exists a triangle $E \to F \to D \to E[1]$ with $E \in \mathrm{Perf}_Z(X)$ and $D \in D_{\mathrm{Qcoh}}(X)^{\leq -2}$. Then, in the triangle $D[-1] \to E \to F \to D$ we have that $D[-1] \in D_{\mathrm{Qcoh}}(X)^{\leq -1}$ and $F \in D_{\mathrm{Qcoh}}(X)^{\geq 0}$. So, this must be the truncation triangle for E with respect to the standard t-structure[13]. Therefore, $F = \tau^{\geq 0}E$ is a perfect complex by our hypothesis. ☐

Now we move on to the more general case, where the two t-structures are not necessarily the same. We begin with the following definition which gives an equivalence relation on the collection of t-structures. We will see that the proof above works even if the two t-structures are merely equivalent. Then, the remaining task would be to show that the two t-structures are equivalent, regardless of the choice of the bounded t-structure on $\mathrm{Perf}_Z(X)$ we begin with.

Definition 1.13 ([Nee24a])**.** Two t-structures $(\mathcal{U}_1, \mathcal{V}_1)$ and $(\mathcal{U}_2, \mathcal{V}_2)$ on a triangulated category $\mathcal{T}$ are *equivalent* if there exists an integer $n \geq 0$ such that $\mathcal{U}_2[n] \subseteq \mathcal{U}_1 \subseteq \mathcal{U}_2[-n]$ (equivalently $\mathcal{V}_2[n] \supseteq \mathcal{V}_1 \supseteq \mathcal{V}_2[-n]$).

We now show that the proof above can be easily modified to the case when the two t-strucutures are equivalent.

Theorem 1.14. *Let X be a Noetherian scheme with a closed subset Z and $(\mathcal{U}, \mathcal{V})$ be a t-structure on $\mathrm{Perf}_Z(X)$ such that the t-structure $(\mathcal{T}^{\leq 0}_{\mathcal{U}}, \mathcal{T}^{\geq 0}_{\mathcal{U}})$ is equivalent to the standard t-structure on $D_{\mathrm{Qcoh},Z}(X)$. Then, the natural inclusion $\mathrm{Perf}_Z(X) \to D^b_{\mathrm{coh},Z}(X)$ is an equivalence.*

Proof. First, note that by the definition of equivalence of t-structures, there exists $N \geq 0$ with

$$D_{\mathrm{Qcoh},Z}(X)^{\leq -N} \subseteq \mathcal{T}^{\leq 0}_{\mathcal{U}} \subseteq D_{\mathrm{Qcoh},Z}(X)^{\leq N}.$$

It is enough to show that every object in $D^b_{\mathrm{coh},Z}(X)$ is a perfect complex. Let $F \in D^b_{\mathrm{coh},Z}(X)$. By replacing F by $F[n]$ for some $n \ll 0$ if necessary, we may assume that $F \in D^b_{\mathrm{coh},Z}(X)^{\geq N} \subseteq \mathcal{T}^{\geq 0}_{\mathcal{U}}$. By Theorem 1.12, there exists a triangle $E \to F \to D \to E[1]$ with $E \in \mathrm{Perf}_Z(X)$ and $D \in D_{\mathrm{Qcoh}}(X)^{\leq -N-2}$. Then, in the triangle $D[-1] \to E \to F \to D$ we have that $D[-1] \in D_{\mathrm{Qcoh}}(X)^{\leq -N-1} \subseteq \mathcal{T}^{\leq -1}_{\mathcal{U}}$ and $F \in D_{\mathrm{Qcoh}}(X)^{\geq 0}$. So, this must be the truncation triangle for E with respect to the t-structure $(\mathcal{T}^{\leq 0}_{\mathcal{U}}, \mathcal{T}^{\geq 0}_{\mathcal{U}})$. By the uniqueness of truncation triangles, F is a perfect complex. ☐

So, all that remains to be shown is that for any bounded t-structure $(\mathcal{U}, \mathcal{V})$ on $\mathrm{Perf}_Z(X)$, the induced t-structure $(\mathcal{T}^{\leq 0}_{\mathcal{U}}, \mathcal{T}^{\geq 0}_{\mathcal{U}})$ on $D_{\mathrm{Qcoh},Z}(X)$ is equivalent to the standard t-structure.

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[13] Recall that for any t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ and for any $T \in \mathcal{T}$, there is a *unique* triangle $A \to T \to B \to A[1]$ with $A \in \mathcal{T}^{\leq 0}$ and $B \in \mathcal{T}^{\geq 0}$.

Preferred equivalence class. On a triangulated category which has a compact generator, we can intrinsically define an equivalence class. Notable examples of triangulated categories with a compact generator are $D_{\mathrm{Qcoh},Z}(X)$ for a Noetherian scheme and $D(R)$ for a ring spectrum. We begin with the definition, and then prove that the two t-structures we are interested in both lie in the preferred equivalence class, hence proving their equivalence.

Definition 1.15 ([Nee24a])**.** Let $\mathcal{T}$ be a triangulated category with a compact generator G. Then, the *preferred equivalence class* of t-structures on $\mathcal{T}$ consists of those t-structures which are equivalent to the t-structure $(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0})$ compactly generated by G. Note that this is independent of the choice of the compact generator G.

We begin with the following (well-known and easy) lemma, whose proof we just sketch. The interested reader can see [Nee24a, Lemma 3.5] for a complete proof.

Lemma 1.16. *Let X be a Noetherian finite-dimensional*[14] *scheme. Then, for any perfect complex $E \in \mathrm{Perf}(X)^{\geq n}$, the dual $\mathrm{R}\mathcal{H}om(E, \mathcal{O}_X) \in \mathrm{Perf}(X)^{\leq -n+\dim X}$.*

Proof Sketch. This statement can be checked affine locally, so without loss of generality we can assume $X = \mathrm{Spec}(R)$ for some Noetherian scheme R of finite Krull dimension. Let P be a projective resolution of E by finitely generated projectives. Then, the required result is clear from the fact that $H^n(\sigma^{\leq n} P)$ has a projective resolution of length less than or equal to $\dim(X)$[15]. □

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[14] Note that this is exactly where finite-dimensionality is required.

[15] This can be shown easily by the Auslander-Buchsbaum formula. More fundamentally, this is a consequence of the finiteness of the *finitistic dimension* of R, an observation that has been vastly generalised [BCM$^+$23].

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