METRICS AND APPROXIMABILITY FOR TRIANGULATED CATEGORIES

1. Overview

Triangulated categories are ubiquitous in algebra, topology, and geometry. Neeman has recently introduced certain metric/analytic techniques for triangulated categories in a series of articles [Nee21b, Nee18, Nee26, BNP23, Nee24, CHNS24, Nee25][1], and used them to settle multiple conjectures concerning triangulated categories arising in algebraic geometry, and to prove other new results[2].

The idea of this mini-course is to give a feeling of these techniques in action, by going through some of the interesting results proven using them. Most of the applications so far are in algebraic geometry, although these techniques are slowly being applied in representation theory and homotopy theory also, see for example [CG24, Mat26a, Mat26b, LS25].

In particular, we will look at the connection between the regularity of a Noetherian scheme with the existence of bounded t-structures and strong generators for the category of perfect complexes. Further, we will discuss some new representability theorems and how to use them to give a unified proof of GAGA theorems in algebraic geometry.

In the second half, we will prove that the derived category $D_{\mathrm{Qcoh}}(X)$ of a Noetherian separated scheme is approximable. We will then move on to the passage between various triangulated subcategories of a weakly approximable triangulated category, which is a vast generalisation of a result by Rickard on derived Morita theory. Finally, we will end with a result on the bijection of semiorthogonal decompositions on various (small) triangulated categories associated to a scheme.

Prerequisites. We will assume familiarity with the basics of triangulated categories, or their enhanced incarnations (stable model/$(\infty,1)$-categories, pretriangulated DG-categories ...). As most of the applications we will discuss are algebro-geometric, we will expect some familiarity with the language of schemes, and with the derived categories and functors associated to them.

2. Course Plan

Week 1-2: Bounded t-structures and regularity. We will discuss Neeman's resolution of the conjecture by Antieau, Gepner, and Heller [AGH19, Conjecture 1.5] on the connection between bounded t-structures and regularity for a Noetherian finite dimensional scheme. We will mostly follow [Nee24]. In particular, we will show where weak approximability and the preferred equivalence class of t-structures comes in. We will define (weakly) approximable triangulated categories and prove that the derived categories of connective DG-algebras, ring spectras etc. are approximable.

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[1] In the order the preprints appeared on arXiv.

[2] See also the survey articles [Nee20, Nee21a, Nee23, CNS25].

Week 3-4: Strong generators and regularity. We will discuss the resolution by Neeman of a conjecture on the relation between existence of strong generators and regularity by Bondal and Van den Bergh [BVdB03]. We will follow the argument in [Nee21b]. In particular, we will define and discuss the proof of compact boundedness, and where approximability comes in. If time permits, we will also prove some results on "descent" of strong generators using ideas from [Aok21, DLM24, DLM25].

Week 4: Representability theorems. We will discuss some new representability theorems proven using the metric techniques and sketch a proof of GAGA using these. Along the way we will also introduce the closure of compact objects. Further, if time permits, we will give a new perspective on how the second part of Neeman's result follows from co-approximability of the mock homotopy category of projectives following [MR25b].

Week 5: Approximability and the passage between triangulated subcategories. Following [Nee21b], we will give a proof of approximability for quasi-projective schemes. Then, we will discuss a derived Morita type results on passage between triangulated subcategories of the derived category of a weakly approximable triangulated category proven in [CHNS24].

Week 6: Semiorthogonal decompositions via metric techniques. Following [MR25a], we will prove a result on the bijection of (left/right) admissible subcategories of various (small) triangulated categories associated to a scheme.

References

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