ON THE INTERACTION BETWEEN SEMIORTHOGONAL DECOMPOSITIONS AND METRIC TECHNIQUES FOR TRIANGULATED CATEGORIES

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Abstract. We provide a method of constructing new semiorthogonal decompositions using metric techniques (à la Neeman). Given a semiorthogonal decomposition on a category with a special kind of metric, called a *compressed metric*, we can construct new semiorthogonal decomposition on a category constructed from the given one using the aforementioned metric. In the algebro-geometric setting, this gives us a way of producing new semiorthogonal decompositions on various small triangulated categories associated to a scheme starting from a given one. The general results in this work are related to the work of Sun–Zhang, while its applications to algebraic geometry are related to the work of Bondarko and Kuznetsov–Shinder.

1. Introduction

Triangulated categories have found important applications in diverse areas of mathematics; from geometry, to topology, to representation theory. Over the years, people have develeoped various tools to study triangulated categories. In this work, we will study the interaction between two different tools; semiorthogonal decompositions, and metric techniques for triangulated categories. Semiorthogonal decompositions allow us to understand a more complicated triangulated category by breaking it up into simpler triangulated categories. They have found many applications, especially in algebraic geometry and representation theory, including applications on the connections between these areas. Bondal and Orlov in [BO95, BO02] began the systematic use of semiorthogonal decompositions to study schemes, although there existed algebro-geometric examples of semiorthogonal decompositions before that, including the ones given by Beilinson and Kapranov's full exceptional collections for Grassmanians in [Beĭ78, Kap88]. There has been a lot of progress since then on the interaction between algebraic geometry and semiorthogonal decompositions, with important applications to birational geometry and the minimal model program, see for example the survey articles [Kuz14, Kuz23].

A much more recent development is the use of certain metric techniques to study triangulated categories. Introduced by Neeman, they have been used to prove an array of diverse results, see for example [Nee21, Nee24, CHNS24, Nee25b]. In particular, these metric techniques were used by Canonaco, Neeman, and Stellari in [CHNS24] to prove that we can pass between various intrinsically defined subcategories of a *weakly approximable triangulated category*, which gives a vast generalisation of Rickard's theorem

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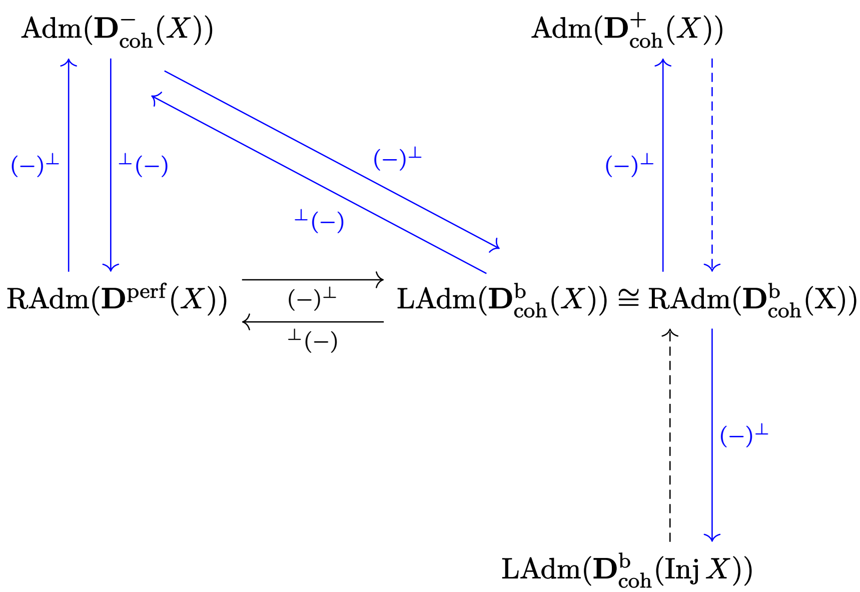
on derived Morita theory from [Ric89]. In the context of algebraic geometry, this implies that various triangulated categories associated to a Noetherian scheme; for example the category of perfect complex, the bounded above/below derived category category of coherent sheaves, etc. determine each other. This immediately begs the question; what triangulated information can be passed amongst these categories? There has been some work related to this question; for t-structures in [BCM+24], for recollements in [SZ21], and for semiorthogonal decompositions in [Bon24].

In this work, we further explore the relation between semiorthogonal decompositions on various essentially small subcategories of a compactly generated triangulated category satisfying certain conditions. In the process of proving the results on semiorthogonal decompositions, we also prove some important results in the emerging theory of metric techniques for triangulated categories. Before we get into the more general results, and what goes into proving them, we first state the implication of the main results to the algebro-geometric setting. Recall that a strictly full triangulated subcategory A of a triangulated category T is left (resp. right) admissible if the inclusion of A in T has a left (resp. right) adjoint. We say A is admissible if it is both left and right admissible. Further, giving a (two component) semiorthogonal decomposition is equivalent to giving a left or a right admissible subcategory. In the result below, $\mathrm{RAdm}(\mathsf{T})$ (resp. $\mathrm{LAdm}(\mathsf{T})$, $\mathrm{Adm}(\mathsf{T})$) will denote the collection of right admissible (resp. left admissible, admissible) subcategories of T.

Theorem A (Theorem 6.3)**.** *Let X be a quasiexcellent scheme which is proper over a Noetherian finite-dimensional ring R. Then, the solid maps in the diagram below exist,*



where $(-)^\perp$ and $^\perp(-)$ by abuse of notation denote the full subcategory of the appropriate category consisting of objects which are left right orthogonal to the given (left and/or right) admissible subcategory respectively.

If we assume that X has a dualizing complex, then the dashed arrows exist too. In either of these cases, whenever arrows exist in both the directions between a pair in the diagram, they give a bijection between the corresponding sets.

Note here that the two arrows in black, and the corresponding bijection, were known before and appear in [Bon24] and in a weaker form in [KS25], while the arrows in blue are new. We should remark here immediately that any scheme of finite type over a field is quasiexcellent, so examples of schemes satisfying the hypothesis of Theorem A are plentiful. The result illustrates concretely the passage of triangulated information amongst various small subcategories of the derived category of a scheme.

If we relax the conditions on the scheme, we lose the bijections in the above result, but we can still get a weaker result. In the following theorem, we summarise the results of this form, including versions for algebras and stacks. The conditions imposed in the theorem below are more stringent than required in some cases for ease of exposition. See the corresponding results in the main body of this work for the general statements. In the statement below, $\operatorname{Coprod}(\mathsf{A})$ will denote the localising subcategory generated by a triangulated subcategory A inside the derived category of quasicoherent sheaves (over the scheme/algebra/stack in question).

Theorem B. *Consider the following data,*

(1) Let X be a Noetherian, separated, finite-dimensional scheme, and $\mathcal{A}$ a coherent $\mathcal{O}_X$-algebra.
(2) Let $\mathcal{X}$ be a concentrated Noetherian stack (concentrated means that the canonical map $\mathcal{X} \to \operatorname{Spec}(\mathbb{Z})$ is concentrated, see [HR17, Definition 2.4]*) such that one of the conditions below hold,*
  - *$\mathcal{X}$ has quasi-finite and separated diagonal.*
  - *$\mathcal{X}$ is a Deligne-Mumford stack of characteristic zero.*

Let $(\mathsf{S}_1, \mathsf{S}_2)$ be any pair as in the table below. Then, for any semiorthogonal decomposition $\langle \mathsf{A}, \mathsf{B} \rangle$ on S_1, we get a semiorthogonal decomposition $\langle \operatorname{Coprod}(\mathsf{A}) \cap \mathsf{S}_2, \operatorname{Coprod}(\mathsf{B}) \cap \mathsf{S}_2 \rangle$ on S_2.

Extra conditions	S_1	S_2	***Reference***
–	$\mathbf{D}^{\mathrm{perf}}(X)$	$\mathbf{D}^{-}_{\mathrm{coh}}(X)$	*Remark 5.9*
B *is admissible*	$\mathbf{D}^{\mathrm{perf}}(X)$	$\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$	*Remark 5.9*
X is J-2, see Definition 4.6	$\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$	$\mathbf{D}^{+}_{\mathrm{coh}}(X)$	*Theorem 5.10*
–	$\mathbf{D}^{\mathrm{perf}}(\mathcal{A})$	$\mathbf{D}^{-}_{\mathrm{coh}}(\mathcal{A})$	*Theorem 5.12*
B *is admissible*	$\mathbf{D}^{\mathrm{perf}}(\mathcal{A})$	$\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A})$	*Theorem 5.12*
X is J-2	$\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A})$	$\mathbf{D}^{+}_{\mathrm{coh}}(\mathcal{A})$	*Theorem 5.13*
X is J-2 & B is admissible	$\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A})$	$\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\operatorname{Inj}\mathcal{A})$	*Theorem 5.13*
–	$\mathbf{D}^{\mathrm{perf}}(\mathcal{X})$	$\mathbf{D}^{-}_{\mathrm{coh}}(\mathcal{X})$	*Theorem 5.14*
B *is admissible*	$\mathbf{D}^{\mathrm{perf}}(\mathcal{X})$	$\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{X})$	*Theorem 5.14*

Note here the generality of this statement; there is no properness assumption on the scheme, and the quasiexcellence is replaced with the weaker J-2 assumption. On the flip side, we have to assume the existence of an extra adjoint in certain cases, which breaks down the bijection of Theorem A.

This result follows from some very general results proved in §3, which should be of significant independent interest, due to their possible applicability beyond the setting of Theorem B. We now get into the details of those results, but again, not in their full generality.

Definition A. Let T be a compactly generated triangulated category with a single compact generator G. Let $({}^{\mathrm{I}}\mathsf{T}_G^{\leq 0}, {}^{\mathrm{I}}\mathsf{T}_G^{\geq 0})$ and $({}^{\mathrm{II}}\mathsf{T}_G^{\geq 0}, {}^{\mathrm{II}}\mathsf{T}_G^{\leq 0})$ denote the t-structure and co-t-structure compactly generated by G respectively, see Definitions 2.6 and 2.8. We define the following full subcategories of T,

- T_c^- is the full subcategory of T consisting of all objects F such that for all $n \in \mathbb{Z}$, there exists a triangle $E_n \to F \to D_n \to \Sigma E_n$ for a compact object E_n and with $D_n \in {}^{\mathrm{I}}\mathsf{T}_G^{\leq -n}$.
- ${}^{\mathrm{I}}\mathsf{T}_c^b = \{F \in \mathsf{T}_c^- : \mathrm{Hom}_{\mathsf{T}}(\Sigma^i G, F) = 0 \text{ for } i \gg 0\}$.
- T_c^+ is the full subcategory of T consisting of all objects F such that for all $n \in \mathbb{Z}$, there exists a triangle $E_n \to F \to D_n \to \Sigma E_n$ for a compact object E_n and with $D_n \in {}^{\mathrm{II}}\mathsf{T}_G^{\geq n}$.
- ${}^{\mathrm{II}}\mathsf{T}_c^b = \{F \in \mathsf{T}_c^+ : \mathrm{Hom}_{\mathsf{T}}(\Sigma^i G, F) = 0 \text{ for } i << 0\}$.

Now, we introduce the following hypotheses.

Hypothesis. Let T be a compactly generated triangulated category with a single compact generator G. Then,

(I) T satisfies Hypothesis (I) if $\mathrm{Hom}_{\mathsf{T}}(\Sigma^{-i} G, G) = 0$ for $i \gg 0$.
(II) T satisfies Hypothesis (II) if $\mathrm{Hom}_{\mathsf{T}}(\Sigma^{i} G, G) = 0$ for $i \gg 0$.

We note here that there are plenty of examples where these hypotheses hold.

Example A. Hypothesis (I) holds for the following categories,

- $\mathbf{D}_{\mathrm{Qcoh}}(X)$ for a Noetherian scheme X.
- $\mathbf{D}_{\mathrm{Qcoh}}(\mathcal{A})$ for a quasicoherent $\mathcal{O}_X$-algebra over a Noetherian scheme X.
- $\mathbf{D}_{\mathrm{Qcoh}}(\mathcal{X})$ for a 1-Thomason stack, see [HR17, Definition 8.1].
- The stable homotopy category $\mathbf{Ho}(\mathrm{Spectra})$.
- The derived category of any connective DG-algebra or a connective $\mathbb{E}_1$-ring spectrum.

Hypothesis (II) holds for the following categories,

- $\mathbf{D}_{\mathrm{Qcoh}}(X)$ for a Noetherian scheme X.
- $\mathbf{K}(\mathrm{Inj}\, X)$ for a Noetherian J-2 scheme X.
- $\mathbf{K}_m(\mathrm{Proj}\, X)$ for a Noetherian, separated, J-2 scheme X, where $\mathbf{K}_m(\mathrm{Proj}\, X)$ denote the mock homotopy category of projectives as defined in [Mur08].
- $\mathbf{K}(\mathrm{Inj}\, \mathcal{A})$ for a quasicoherent $\mathcal{O}_X$-algebra over a Noetherian J-2 scheme X.
- The derived category of any co-connective DG-algebra or a co-connective $\mathbb{E}_1$-ring spectrum.

With these hypotheses, we have the following general results on producing semiorthogonal decompositions, where Coprod(A) denotes the localising subcategory generated by the triangulated subcategory A of T.

Theorem C (Corollaries 3.6, 3.7, 3.14 and 3.15)**.** *Let* T *be a triangulated category.*

(1) Assume T *satisfies Hypothesis (I) and let* $\langle \mathsf{A}, \mathsf{B} \rangle$ *be a semiorthogonal decomposition on* T^c*. Then* $\langle \mathrm{Coprod}(\mathsf{A}) \cap \mathsf{T}^-_c, \mathrm{Coprod}(\mathsf{B}) \cap \mathsf{T}^-_c \rangle$ *is a semiorthogonal decomposition on* T^-_c*. If* B *is further assumed to be admissible, then* $\langle \mathrm{Coprod}(\mathsf{A}) \cap {}^{\mathrm{I}}\mathsf{T}^b_c, \mathrm{Coprod}(\mathsf{B}) \cap {}^{\mathrm{I}}\mathsf{T}^b_c \rangle$ *is a semiorthogonal decomposition on* ${}^{\mathrm{I}}\mathsf{T}^b_c$*.*

(2) Assume T *satisfies Hypothesis (II) and let* $\langle \mathsf{A}, \mathsf{B} \rangle$ *be a semiorthogonal decomposition on* T^c*. Then,* $\langle \mathrm{Coprod}(\mathsf{A}) \cap \mathsf{T}^+_c, \mathrm{Coprod}(\mathsf{B}) \cap \mathsf{T}^+_c \rangle$ *is a semiorthogonal decomposition on* T^+_c*. If* B *is further assumed to be admissible, then* $\langle \mathrm{Coprod}(\mathsf{A}) \cap {}^{\mathrm{II}}\mathsf{T}^b_c, \mathrm{Coprod}(\mathsf{B}) \cap {}^{\mathrm{II}}\mathsf{T}^b_c \rangle$ *is a semiorthogonal decomposition on* ${}^{\mathrm{II}}\mathsf{T}^b_c$ *.*

Therefore, in order to obtain Theorem B from Theorem C, we need to compute the subcategories defined in Definition A in the context of Theorem B. It turns out that some of the work has already been done for us. For the examples related to Hypothesis (I), the computation of T^-_c and ${}^{\mathrm{I}}\mathsf{T}^b_c$ follows from [Nee24, DLMR24, HLLP25, DLMRP25], see Propositions 5.5 to 5.7.

For the examples related to Hypothesis (II), more work is required to compute T^+_c and ${}^{\mathrm{II}}\mathsf{T}^b_c$. The main issue is that a priori there is not much control on the co-t-structure compactly generated by a compact object which is used to define these categories, see Definition A. Let us take the case of $\mathsf{T} = \mathbf{K}(\mathrm{Inj}\, X)$ for a Noetherian J-2 scheme X to illustrates our main strategy to deal with this issue. $\mathbf{K}(\mathrm{Inj}\, X)$ has a standard co-t-structure, given by the brutal truncation of complexes. This co-t-structure is much easier to work with, because of the simple form of the truncations. Hence, what we need is to connect these two co-t-structures. One way to do this is through the theory of co-approximable triangulated categories, as introduced in [MR25]. This notion is similar to that of weakly approximable triangulated categories à la Neeman, the main difference being that we work with co-t-structures instead of t-structures. We prove that the homotopy category of injectives associated to a large class of schemes, and algebras over schemes, is weakly co-approximable *using the co-t-structure defined via the brutal truncations*, see Theorems 4.13 and 4.14. Similar result also hold for the mock homotopy category of projectives, see Proposition 4.15. These results help us in relating the two co-t-structures. Further, these examples of weakly co-approximable triangulated categories should be of independent interest with the growing theory surrounding approximable triangulated categories, and in general metric techniques for triangulated categories.

Comparison with the literature. We owe the reader a comparison with similar results in the existing literature, especially the results in [Bon24, KS25]. [Bon24, Theorem 1.3] gives us the bijection given by the arrows in black in Theorem A. Note that the condition on the scheme in our result is slightly more general, but the same generality can be achieved by the proof in [Bon24] using [Aok21, Main Theorem]. [KS25, Corollary 6.5] achieves a similar result, although under much more stringent conditions on the scheme. The rest of the arrows in Theorem A do not appear in these works, but the reader is encouraged to look at the various results in similar spirit in [Bon24]. The results in both of these papers assume that the scheme is proper, and hence it does not make much sense to compare them with Theorem B.

Finally, we would be remiss not to mention [SZ21], which proves a result similar to Theorem C, and its more general version Theorem 3.13. To compare the results, note that an admissible subcategory gives a recollement, and that T^b_c is the completion of T^c if for example T is weakly $\mathcal{G}$-approximable by [MR25, Corollary 8.4]. [SZ21, Theorem 1.2] has much weaker conditions on the category, but assumes that all the functors involved in the recollement are compression functors, which is not obvious if we are in the setting of Theorem 3.13. Further, Theorem 3.13 gives a very concrete description of the new right admissible subcategory we get, which again is not obvious to get from [SZ21, Theorem 1.2].

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2. Background and notation

Throughout this work, we will use Σ to denote the shift functor in the definition of triangulated categories. For any functor $F : \mathsf{S} \to \mathsf{T}$, $F(\mathsf{S})$ will denote the essential image of S under F. Further, all the gradings involved in this work will be cohomological. For full subcategories A and C of a triangulated category T, $\mathsf{A} \star \mathsf{C}$ will denote the full subcategory consisting of objects B such that there exists a triangle $A \to B \to C \to \Sigma A$ with $A \in \mathsf{A}$ and $C \in \mathsf{C}$. For any full subcategory A of T, $\mathsf{A}^\perp$ (resp. ${}^\perp\mathsf{A}$) will denote the full subcategory of all objects B such that $\mathrm{Hom}_\mathsf{T}(A, B) = 0$ (resp. $\mathrm{Hom}_\mathsf{T}(B, A) = 0$) for all $A \in \mathsf{A}$. We will say that a triangulated category T **has coproducts** if all set-indexed coproducts of objects of T exist in T. Further, for a triangulated category with coproducts, we say a subcategory is **closed under coproducts** if it is closed under set-indexed coproducts. Finally, for a triangulated category T with coproducts, we denote the full subcategory of compact objects by T^c.

We now recall some background material related to metrics on triangulated categories.

Metrics, generating sequences, and weak co-approximability. We begin with some notation from [BvdB03, Nee25b].

Definition 2.1. Let C be a full subcategory of a triangulated category T. Then,

- $\mathrm{smd}(\mathsf{C})$ denotes the closure of C under summands.
- $\mathrm{coprod}(\mathsf{C})$ denotes the closure of C under finite coproducts and extensions.
- If T has coproducts, then $\mathrm{Coprod}(\mathsf{C})$ denotes the closure of C under coproducts and extensions. Note that if $\mathsf{C} = \Sigma\mathsf{C}$, then $\mathrm{Coprod}(\mathsf{C})$ is the localising subcategory of T generated by C.

- Let G be an object of T. Then, for any $a \leq b$, we define,
$$\langle G\rangle^{[a,b]} := \text{smd}(\text{coprod}(\{\Sigma^{-i}G : a \leq i \leq b\}))$$
We also extend this definition for $a = -\infty$ and $b = \infty$ in the obvious way.
- We define $\langle G\rangle := \bigcup_{n\geq 0}\langle G\rangle^{[-n,n]}$. This is in fact the thick subcategory generated by G. If G is a compact generator for a triangulated category T, then the subcategory of compacts $\mathsf{T}^{\mathrm{c}} = \langle G\rangle$.
- Let G be an object of T. If T has coproducts, we define,
$$\overline{\langle G\rangle}^{[a,b]} := \text{smd}(\text{Coprod}(\{\Sigma^{-i}G : a \leq i \leq b\}))$$
We also extend this definition for $a = -\infty$ and $b = \infty$ in the obvious way.

We begin by stating the definition of an extended good metric for a triangulated category, which is just the $\mathbb{Z}$-graded version of Neeman's notion of a good metric, see [Nee20, Definition 10] and [MR25, Definition 3.1].

Definition 2.2. Let T be a triangulated category.

- An **extended good metric** is a decreasing sequence of strictly full subcategories $\{\mathcal{M}_n\}_{n\in\mathbb{Z}}$ each closed under extensions and containing 0 such that,
$$\Sigma^{-1}\mathcal{M}_{n+1} \cup \mathcal{M}_{n+1}\Sigma\mathcal{M}_{n+1} \subseteq \mathcal{M}_n$$
for all integers n.
- We say an extended good metric $\{\mathcal{R}_n\}_{n\in\mathbb{Z}}$ is an **orthogonal metric** if ${}^{\perp}[(\mathcal{R}_n)^{\perp}] = \mathcal{R}_n$ for all integers n.
- Finally, we say two metrics $\mathcal{M}$ and $\mathcal{N}$ are **equivalent** if for all i, there exists $j \geq 0$ such that $\mathcal{M}_{i+j} \subseteq \mathcal{N}_j \subseteq \mathcal{M}_{i-j}$, and we say that they are $\mathbb{N}$**-equivalent** if the j can be chosen independent of i.

From now on, metric would always mean an extended good metric unless explicitly stated otherwise.

We now recall a few notions from [MR25].

Definition 2.3. Let T be a compactly generated triangulated category. Then,

- A **pre-generating sequence** is a sequence $\{\mathcal{G}[n,n]\}_{n\in\mathbb{Z}}$ of full subcategories of compact objects such that $\text{smd}(\text{coprod}(\bigcup_{n\in\mathbb{Z}}\mathcal{G}[n,n]))) = \mathsf{T}^{\mathrm{c}}$. Given a pre-generating sequence $\mathcal{G}$, and $a \leq b$, we define the full subcategory,
$$\mathcal{G}^{[a,b]} := \text{smd}\left(\text{coprod}\left(\bigcup_{i=a}^{\mathrm{b}}\mathcal{G}[i,i]\right)\right)$$
We also extend this definition for $a = -\infty$ and $b = \infty$ in the obvious way.
- A pre-generating sequence $\mathcal{G}$ is a **generating sequence** if
$$\Sigma^{-1}\mathcal{G}[n,n] \cup \mathcal{G}[n,n] \cup \Sigma\mathcal{G}[n,n] \subseteq \mathcal{G}^{[n-1,n+1]}$$
for all $n \in \mathbb{Z}$. It is further a **finite generating sequence** if $\mathcal{G}[n,n]$ consists of finitely many objects for each integer n.

- Given a generating sequence $\mathcal{G}$, we can define an extended good metric $\mathcal{M}^{\mathcal{G}}$ on T^c and an orthogonal metric $\mathcal{R}^{\mathcal{G}}$ on T defined by,
$$\mathcal{M}^{\mathcal{G}}_n := \mathcal{G}^{(-\infty,-n]} = \operatorname{smd}\Big(\operatorname{coprod}\Big(\bigcup_{i\leq -n}\mathcal{G}[i,i]\Big)\Big), \quad \mathcal{R}^{\mathcal{G}}_n := {}^{\perp}[(\mathcal{M}^{\mathcal{G}}_n)^{\perp}]$$
for all $n \in \mathbb{Z}$.

Definition 2.4. Given a pre-generating sequence $\{\mathcal{G}[n,n]\}_{n\in\mathbb{Z}}$, we define its **inverse pre-generating sequence** $\check{\mathcal{G}}$ by setting $\check{\mathcal{G}}[n,n] := \mathcal{G}[-n,-n]$. Note that if $\mathcal{G}$ is a generating sequence, then so is $\check{\mathcal{G}}$.

Example 2.5. Let T be a compactly generated triangulated category with a single compact generator G. In this case we can define the following two finite generating sequences (Definition 2.3),

- $\mathcal{G}[n,n] := \{\Sigma^{-n}G\}$. In this case $\mathcal{M}^{\mathcal{G}}_n = \langle G\rangle^{(-\infty,-n]}$, see Definition 2.1. Further, $\mathcal{R}^{\mathcal{G}} = \mathsf{T}^{\leq -n}_G$ where $(\mathsf{T}^{\leq 0}_G, \mathsf{T}^{\geq 0}_G)$ is the t-structure generated by G, see Definition 2.6. Note that the metric $\mathcal{R}^{\mathcal{G}} \cap \mathsf{T}^c$ is equal to the metric $\mathcal{M}^{\mathcal{G}}$ by the remark at the end of Definition 2.6.
- $\mathcal{G}[n,n] := \{\Sigma^{n}G\}$. In this case $\mathcal{M}^{\mathcal{G}}_n = \langle G\rangle^{[n,\infty)}$, see Definition 2.1. Further, $\mathcal{R}^{\mathcal{G}} = \Sigma^{-n}\mathsf{U}_G$ where $(\mathsf{U}_G, \mathsf{V}_G)$ is the co-t-structure generated by G, see Definition 2.8. Note that the metric $\mathcal{R}^{\mathcal{G}} \cap \mathsf{T}^c$ is equivalent to the metric $\mathcal{M}^{\mathcal{G}}$ by the remark at the end of Definition 2.8.

We now recall the definitions of a (compactly generated) t-structure and a (compactly generated) co-t-structure.

Definition 2.6. [BBD82, Définition 1.3.1] A **t-structure** on a triangulated category T is a pair $(\mathsf{T}^{\leq 0}, \mathsf{T}^{\geq 0})$ of strictly full subcategories such that,

- $\Sigma\mathsf{T}^{\leq 0} \subseteq \mathsf{T}^{\leq 0}$, $\Sigma^{-1}\mathsf{T}^{\geq 0} \subseteq \mathsf{T}^{\geq 0}$, and $\operatorname{Hom}_{\mathsf{T}}(\Sigma\mathsf{T}^{\leq 0}, \mathsf{T}^{\geq 0}) = 0$. We define $\mathsf{T}^{\leq n} := \Sigma^{-n}\mathsf{T}^{\leq 0}$ and $\mathsf{T}^{\geq n} := \Sigma^{-n}\mathsf{T}^{\geq 0}$.
- For all $Y \in \mathsf{T}$, there exists a triangle $X \to Y \to Z \to \Sigma X$ with $X \in \Sigma\mathsf{T}^{\leq 0}$ and $Z \in \mathsf{T}^{\geq 0}$.

Given a t-structure, we get an orthogonal metric associated to it on T which is given by $\{\mathsf{T}^{\leq -n}\}_{n\in\mathbb{Z}}$. Further, if T has coproducts, we also get a metric associated to it on T^c which is given by $\{\mathsf{T}^{\leq -n} \cap \mathsf{T}^c\}_{n\in\mathbb{Z}}$.

If T has coproducts, then for any compact object $G \in \mathsf{T}^c$, $(\mathsf{T}^{\leq 0}_G, \mathsf{T}^{\geq 0}_G)$ is a t-structure by [ATJLSS03, Theorem A.1] where,
$$\mathsf{T}^{\leq 0}_G := \operatorname{Coprod}(\{\Sigma^i G : i \geq 0\}), \mathsf{T}^{\geq 0}_G := (\Sigma\mathsf{T}^{\leq 0}_G)^{\perp}$$
This is said to be the **t-structure generated by** G. We note here that $\mathsf{T}^{\leq 0}_G \cap \mathsf{T}^c = \langle G\rangle^{(-\infty,0]}$, which implies that the metric associated to it on T^c is the same as the metric $\{\langle G\rangle^{(-\infty,-n]}\}_{n\in\mathbb{Z}}$.

Remark 2.7. Let T be a triangulated category.

- A strictly full subcategory U is called a **aisle** if $\Sigma\mathsf{U} \subseteq \mathsf{U}$, and the inclusion $\mathsf{U} \to \mathsf{T}$ has a right adjoint.
- A strictly full subcategory U is called a **coaisle** if $\Sigma^{-1}\mathsf{V} \subseteq \mathsf{V}$, and the inclusion $\mathsf{V} \to \mathsf{T}$ has a left adjoint.

Then, we have the following,

- If (U,V) is a t-structure on T then U is an aisle and V is a coaisle.
- If U is an aisle, then, $(\mathsf{U},\Sigma\mathsf{U}^{\perp})$ is a t-structure on T.
- If V is an coaisle, then, $(\Sigma^{-1}({}^{\perp}\mathsf{V}),\mathsf{V})$ is a t-structure on T.

Definition 2.8 ([Bon10, Definition 1.1.1] and [Pau08, Definition 1.4])**.** A **co-t-structure** on a triangulated category T is a pair $(\mathsf{T}^{\geq 0},\mathsf{T}^{\leq 0})$ of strictly full subcategories such that,

- $\Sigma\mathsf{T}^{\leq 0}\subseteq\mathsf{T}^{\leq 0}$, $\Sigma^{-1}\mathsf{T}^{\geq 0}\subseteq\mathsf{T}^{\geq 0}$, and $\mathrm{Hom}_{\mathsf{T}}(\mathsf{T}^{\geq 0},\Sigma\mathsf{T}^{\leq 0})=0$. We define $\mathsf{T}^{\leq n}:=\Sigma^{-n}\mathsf{T}^{\leq 0}$ and $\mathsf{T}^{\geq n}:=\Sigma^{-n}\mathsf{T}^{\geq 0}$.
- For all $Y\in\mathsf{T}$, there exists a triangle $X\to Y\to Z\to\Sigma X$ with $Z\in\Sigma\mathsf{T}^{\leq 0}$ and $X\in\mathsf{T}^{\geq 0}$.

Given a co-t-structure, we get an orthogonal metric associated to it on T which is given by $\{\mathsf{T}^{\geq n}\}_{n\in\mathbb{Z}}$. Further, if T has coproducts, we also get a metric associated to it on T^c which is given by $\{\mathsf{T}^{\geq n}\cap\mathsf{T}^c\}_{n\in\mathbb{Z}}$.

If T has coproducts, then for any compact object $G\in\mathsf{T}^c$, $(\mathsf{U}_G,\mathsf{V}_G)$ is a co-t-structure by [Pau12, Theorem 5] where,

$$\mathsf{V}_G:=(\{\Sigma^{-i}G:i\geq 1\})^{\perp},\mathsf{U}_G:={}^{\perp}(\Sigma\mathsf{V}_G)$$

This is said to be the **co-t-structure generated by** G. We note here that $\mathsf{U}_G^{\leq 0}\cap\mathsf{T}^c\subseteq\langle G\rangle^{[0,\infty)}$ by [Bon22, Theorem 2.3.4], see Definition 2.1. This implies that the metric associated to it on T^c is the $\mathbb{N}$-equivalent to the metric $\{\langle G\rangle^{[n,\infty)}\}_{n\in\mathbb{Z}}$, see Definition 2.2.

Definition 2.9. For a triangulated category T, two t-structures (resp. co-t-structures) are said to be equivalent if the corresponding metric on T are equivalent, see Definitions 2.2, 2.6 and 2.8. Note that this is the same as the metrics being $\mathbb{N}$-equivalent. Further, let T be a compactly generated triangulated category with a compact generator G. Then, we say a t-structure (resp. co-t-structure) (U,V) lies in the

- **preferred quasiequivalence class of t-structures (resp. co-t-structures)** if the metric corresponding it to it on T^c is equivalent to the metric corresponding to the t-structure (resp. co-t-structure) generated by G, see Definition 2.6 (resp. Definition 2.8). Note that this is the same as the existence of a positive integer n such that $\langle G\rangle^{(-\infty,-n]}\subseteq\mathsf{U}\cap\mathsf{T}^c\subseteq\langle G\rangle^{(-\infty,n]}$ (resp. $\langle G\rangle^{[n,\infty)}\subseteq\mathsf{U}\cap\mathsf{T}^c\subseteq\langle G\rangle^{[-n,\infty)}$), see Definition 2.1.
- **preferred equivalence class of t-structures (resp. co-t-structures)** if the metric corresponding it to it on T is equivalent to the orthogonal metric corresponding to the t-structure (resp. co-t-structure) generated by G, see Definition 2.6 (resp. Definition 2.8).

We now define the closure of the compacts in the specific context we are interested in. For a more general definition, see [MR25, Definition 3.17].

Definition 2.10. Let T be a compactly generated triangulated category with a generating sequence $\mathcal{G}$, and with an orthogonal metric $\mathcal{R}$ on T, see Definitions 2.2 and 2.3. Then, we define,

- $\overline{\mathsf{T}^c}:=\bigcap_{n\in\mathbb{Z}}\mathsf{T}^c\star\mathcal{R}_n$, that is, $\overline{\mathsf{T}^c}$ is the full subcategory of T consisting of all objects F such that for all $n\in\mathbb{Z}$, there exists a triangle $E_n\to F\to D_n\to\Sigma E_n$ with $E_n\in\mathsf{T}^c$ and $D_n\in\mathcal{R}_n$. We call $\overline{\mathsf{T}^c}$ the **closure of the compacts**.

- $\mathcal{G}^{\perp} := \bigcup_{n\in\mathbb{Z}}(\mathcal{M}^{\mathcal{G}}_n)^{\perp}$, see Definition 2.3 for the definition of $\mathcal{M}^{\mathcal{G}}_n$.
- $\mathsf{T}^{b}_{c} := \overline{\mathsf{T}^{c}} \cap \mathcal{G}^{\perp}$. We call T^{b}_{c} the full subcategory of **the bounded objects in the closure of the compacts**.

Note that these notions only depends on the equivalence class (Definition 2.2) of the metric $\mathcal{R}$. If we are given a triangulated category with a generating sequence, by default we will compute the closure of the compacts with respect to the orthogonal metric $\mathcal{R}^{\mathcal{G}}$ (Definition 2.3) unless explicitly stated otherwise.

The following lemma follows from [MR25, Lemma 6.2], but we give a sketch of the proof for the sake of completeness.

Lemma 2.11. *Let T be a compactly generated triangulated category with a generating sequence $\mathcal{G}$ (Definition 2.3), and with an orthogonal metric $\mathcal{R}$ (Definition 2.2) on T such that for all n, $\mathrm{Hom}_{\mathsf{T}}(\mathcal{G}[n,n],\mathcal{R}_i) = 0$ for $i \gg 0$. Then, an object F lies in the closure of the compacts $\overline{\mathsf{T}^{c}}$, if and only if there exists a sequence $E_1 \to E_2 \to E_3 \to E_4 \to \cdots$ in T^{c} mapping to F such that $\mathrm{Cone}(E_n \to F) \in \mathcal{R}_n$. Furthermore, for any such sequence, $\underrightarrow{\mathrm{Hocolim}}\, E_i \to F$ is an isomorphism.*

Proof. If an object F has such a sequence mapping to it, then by Definition 2.10 it lies in the closure of the compacts.

We now prove the converse statement. First, note that as $\mathcal{G}$ is a generating sequence, $\mathsf{T}^{c} = \bigcup_{n\geq 0}\mathcal{G}^{[-n,n]}$, see Definition 2.3. So, by the hypothesis, we get that for any $H \in \mathsf{T}^{c}$, $\mathrm{Hom}_{\mathsf{T}}(H,\mathcal{R}_i) = 0$ for $i \gg 0$. Using this, it is easy to show that the closure of the compacts $\overline{\mathsf{T}^{c}}$ (Definition 2.10) is triangulated. Now, let $F \in \overline{\mathsf{T}^{c}}$. We will construct the required sequence inductively. The base case holds by Definition 2.10. So, we just need to show the inductive step. Suppose we have the sequence $E_1 \to \cdots \to E_n$ satisfying the required properties. If $D_n := \mathrm{Cone}(E_n \to F)$, then D_n lies in $\mathcal{R}_n \cap \overline{\mathsf{T}^{c}}$. By the argument at the beginning of this proof, there exists an integer n_1, which we can choose so that $n_1 > n$, such that $\mathrm{Hom}_{\mathsf{T}}(E_n,\mathcal{R}_i) = 0$ for all $i \geq n_1$. As $D_n \in \overline{\mathsf{T}^{c}}$, there exists a triangle $\widetilde{E}_n \to D_n \to D_{n+1} \to \Sigma\widetilde{E}_n$ with $\widetilde{E}_n \in \mathsf{T}^{c}$ and $D_{n+1} \in \mathcal{R}_{n_1}$. Applying the octahedral axiom to the composible morphisms $F \to D_n \to D_{n+1}$ gives us the required object $E_{n+1} \in \mathsf{T}^{c}$, and the triangle $E_{n+1} \to F \to D_{n+1} \to \Sigma E_{n+1}$.

By the previous paragraph, we have that for any compact object H, $\mathrm{Hom}_{\mathsf{T}}(H,\mathcal{R}_i) = 0$ for $i \gg 0$. And so, $\mathrm{Hom}_{\mathsf{T}}(H,-)$ sends $E_n \to F$ to an isomorphism for large enough n. Let $E = \underrightarrow{\mathrm{Hocolim}}\, E_i$. We have a map $E \to F$ as the sequence maps to F. Then $D := \mathrm{Cone}(E \to F)$ lies in $\mathcal{R}_n$ for all n by the construction of the sequence E_*. So, combined with the observation at the beginning of this paragraph, we have that for any $H \in \mathsf{T}^{c}$, $\mathrm{Hom}_{\mathsf{T}}(H,D) = 0$, and therefore the functor $\mathrm{Hom}_{\mathsf{T}}(H,-)$ sends $E \to F$ to an isomorphism. As T^{c} is compactly generated, this implies that the map $E \to F$ is an isomorphism. $\square$

We will have special notation for the subcategories defined in Definition 2.10 in some cases, partly to be consistent with the notation in the literature, and partly to highlight these special cases as they will be the most important examples for the purpose of applications.

Convention 2.12. Let T be a compactly generated triangulated category with a single compact generator G.

- Suppose we equip T with the generating sequence given by $\mathcal{G}[n,n] = \{\Sigma^{-n}G\}$ and with the orthogonal metric corresponding to some t-structure on T, see Definition 2.6. Then, the closure of the compacts is denote by T^-_c and $\mathcal{G}^\perp$ is denoted by T^+.
- Suppose we equip T with the generating sequence given by $\mathcal{G}[n,n] = \{\Sigma^{n}G\}$ and with the orthogonal metric corresponding to some co-t-structure on T, see Definition 2.8. Then, the closure of the compacts is denote by T^+_c and $\mathcal{G}^\perp$ is denoted by T^-.

We will mostly be working with t-structures (resp. co-t-structures) lying in the preferred quasiequivalence class, see Definition 2.9. Note that the closure of the compacts will be same if computed with respect to any metric coming from a t-strcucture (resp. co-t-structure) in the preferred quasiequivalence class.

Finally, we state the definition of weak co-approximability, see [MR25, Definitions 7.8 and 7.9]

Definition 2.13. Let T be a compactly generated triangulated category with a single compact generator G and a co-t-structure (U,V) such that $\mathrm{Hom}_{\mathsf{T}}(G,\Sigma^{-n}\mathsf{U}) = 0$ and $\Sigma^{-n}G \in \mathsf{U}$ for $n \gg 0$. Then,

- T is **weakly co-approximable** if there exists $N \geq 0$ such that for all $F \in \mathsf{U}$, there exists a triangle $E \to F \to D \to \Sigma F$ with $D \in \Sigma^{-1}\mathsf{U}$ and $E \in \overline{\langle G\rangle}^{[-N,N]}$, see Definition 2.1.
- T is **weakly co-quasiapproximable** if there exists $N \geq 0$ such that for all $F \in \mathsf{U}\cap\mathsf{T}^+_c$, there exists a triangle $E \to F \to D \to \Sigma F$ with $D \in \Sigma^{-1}\mathsf{U}\cap\mathsf{T}^+_c$ and $E \in \overline{\langle G\rangle}^{[-N,N]}$, where we define T^+_c with respect to the given co-t-structure, see Convention 2.12.

Convention 2.14. For the sake of making statements less cumbersome, from now onward we will sometimes abbreviate weakly co-approximable (resp. weak co-approximability) to weakly co-approx (resp. weak co-approx). Similarly, we will abbreviate weakly co-quasiapproximable (resp. weak co-approximability) to weakly co-quasiapprox (resp. weak co-quasiapprox).

Although the following result also does follow from the work in [MR25], we give a proof for the sake of completeness.

Lemma 2.15. *Let T be a weakly co-approx (resp. co-quasiapprox) triangulated category with a co-t-structure (U,V) as in the definition of weak co-approx (resp. co-quasiapprox). Then, (U,V) lies in the preferred quasiequivalence class of co-t-structures on T, see Definition 2.9.*

Proof. Let $F \in \mathsf{U}\cap\mathsf{T}^+_c$. We begin by inductively producing a sequence $E_1 \to E_2 \to E_3 \to E_4 \to \cdots$ mapping to F such that for all integers n, $E_n \in \overline{\langle G\rangle}^{[-N,\infty)}$ and $\mathrm{Cone}(E_n \to F) \in \Sigma^{-n}\mathsf{U}$ (resp. $\mathrm{Cone}(E_n \to F) \in \Sigma^{-n}\mathsf{U}\cap\mathsf{T}^+_c$) for the integer N as in the definition of weak co-approx (resp. co-quasiapprox). The base case holds by Definition 2.13. So, we just need to show the inductive step. Suppose we have the sequence $E_1 \to \cdots \to E_n$ satisfying the required properties. If $D_n := \mathrm{Cone}(E_n \to F)$, then $\Sigma^n D_n$ lies in U (resp. $\mathsf{U}\cap\mathsf{T}^+_c$). So, by Definition 2.13, we get a triangle $\widetilde{E}_n \to D_n \to$

$D_{n+1} \to \widetilde{E}_n$ with $\widetilde{E}_n \in \overline{\langle G\rangle}^{[-N,\infty)}$ and $D_{n+1} \in \Sigma^{-n-1}\mathsf{U}$ (resp. $D_{n+1} \in \Sigma^{-n-1}\mathsf{U} \cap \mathsf{T}_c^+$). By applying the octahedral axiom to $F \to D_n \to D_{n+1}$, we get the required triangle $E_{n+1} \to F \to D_{n+1} \to \Sigma E_{n+1}$ and a map $E_n \to E_{n+1}$ factoring $E_n \to F$.

Let $E = \underrightarrow{\operatorname{Hocolim}} E_i$. We have a map $E \to F$ as the sequence maps to F, and let $D := \operatorname{Cone}(E \to F)$. Then, we get the following 3×3 diagram from the top left commutative square,

$$\begin{array}{ccccc}
\bigoplus_i E_i & \xrightarrow{1-\text{shift}} & \bigoplus_i E_i & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_i F & \xrightarrow{1-\text{shift}} & \bigoplus_i F & \longrightarrow & F \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_i D_i & \dashrightarrow & \bigoplus_i D_i & \dashrightarrow & D
\end{array}$$

which implies that $D \in \Sigma^{-n}\mathsf{U}$ for all $n \geq 1$. But, by Definition 2.13, for any integer i, $\operatorname{Hom}_{\mathsf{T}}(\Sigma^i G, \Sigma^{-n}\mathsf{U}) = 0$ for $n \gg 0$. Therefore for all $i \in \mathbb{Z}$, $\operatorname{Hom}_{\mathsf{T}}(\Sigma^i G, D) = 0$ and hence $\operatorname{Hom}_{\mathsf{T}}(\Sigma^i G, -)$ sends $E \to F$ to an isomorphism. As G is a compact generator, this implies that the map $E \to F$ is an isomorphism.

Finally, as $F \cong E = \underrightarrow{\operatorname{Hocolim}} E_n$ and $E_n \in \overline{\langle G\rangle}^{[-N,\infty)}$ for all n, we get that $F \in \overline{\langle G\rangle}^{[-N-1,\infty)}$, and hence $\mathsf{U} \cap \mathsf{T}_c^+ \subseteq \overline{\langle G\rangle}^{[-N-1,\infty)}$. By [Nee21, Proposition 1.9], this implies that $\mathsf{U} \cap \mathsf{T}^c \subseteq \langle G\rangle^{[-N-1,\infty)}$. Conversely, by Definition 2.13, $\Sigma^{-i}G \in \mathsf{U}$ for $i \gg 0$, and hence there exists $n \geq 0$ such that $\langle G\rangle^{[-n,\infty)} \subseteq \mathsf{U} \cap \mathsf{T}^c$, which completes the proof. □

We end this section by some recollection of standard facts and definitions related to admissible subcategories, semiorthogonal decompositions, and localisation sequences.

Localisations, recollements, admissible subcategories, and semiorthogonal decompositions. In this subsection, we recall some concepts related to localisations of triangulated categories, and the existence of adjoints. We begin with the following well known result. For a reference, see [Nee01, Chapter 9, page 311-318].

Theorem 2.16. *Let T be a triangulated category, and $i_* : \mathsf{U} \to \mathsf{T}$ a fully faithful functor. Let the Verdier quotient of T by the essential image of i_* be $\mathsf{V} := \mathsf{T}/i_*(\mathsf{U})$, with the localisation functor $j^* : \mathsf{T} \to \mathsf{V}$. Then,*

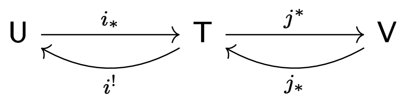
- *i_* has a right adjoint $i^!$ if and only if j^* has right adjoint j_*. Further, if the right adjoints exist, then j_* is fully faithful, and $i^!$ is a Verdier localisation functor, identifying U with $\mathsf{T}/j_*(\mathsf{V})$.*
- *i_* has a left adjoint i^* if and only if j^* has a left adjoint $j_!$. Further, if the left adjoints exist, then $j_!$ is fully faithful, and i^* is a Verdier localisation functor, identifying U with $\mathsf{T}/j_!(\mathsf{V})$.*

Definition 2.17. Let U, T and V be a triangulated category with functors,

$$\mathsf{U} \xrightarrow{i_*} \mathsf{T} \xrightarrow{j^*} \mathsf{V} \tag{2.1}$$

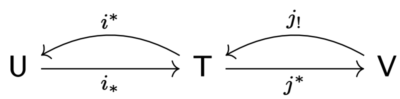
with i_* a fully faithful functor and j^* a Verdier localisation functor, identifying V with $\mathsf{T}/i_*(\mathsf{U})$. Then, we say,

- (2.1) is a **localisation sequence** if i_* (or, equivalently j^*) has a right adjoint. This gives us the diagram,



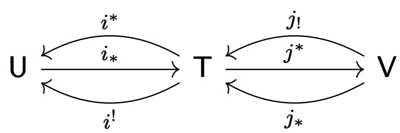
For any $t \in \mathsf{T}$, we get a triangle $i_*i^!t \to t \to j_*j^*t \to \Sigma i_*i^!t$ which show that $(i_*\mathsf{U}, \Sigma j_*\mathsf{V})$ is both a t-structure and a co-t-structure on T.

- (2.1) is a **colocalisation sequence** if i_* (or, equivalently j^*) has a left adjoint. This gives us the diagram,



For any $t \in \mathsf{T}$, we get a triangle, $j_!j^*t \to t \to i_*i^*t \to \Sigma j_!j^*t$ which show that $(j_!\mathsf{V}, \Sigma i_*\mathsf{U})$ is both a t-structure and a co-t-structure on T.

- (2.1) is a recollement if it is both a localising and a colocalising sequence. We get the following diagram in such a case,

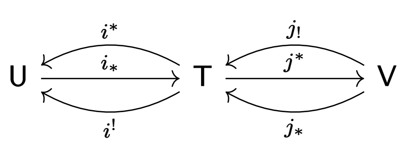


with each functor left adjoint to the one beneath it.

We now define admissible subcategories.

Definition 2.18. Let T be a triangulated category with a strictly full triangulated subcategory U. Then,

- U is **right admissible** if it is an aisle, see Remark 2.7. Note that this gives us the localisation sequence $\mathsf{U} \to \mathsf{T} \to \mathsf{U}^\perp$. Conversely, given a localisation sequence $\mathsf{U} \xrightarrow{i_*} \mathsf{T} \xrightarrow{j^*} \mathsf{V}$, the essential image of i_* is a right admissible category.
- U is **left admissible** if it is a coaisle, see Remark 2.7. Note that this gives us the localisation sequence ${}^\perp\mathsf{U} \to \mathsf{T} \to \mathsf{U}$. Conversely, given a localisation sequence $\mathsf{U} \xrightarrow{i_*} \mathsf{T} \xrightarrow{j^*} \mathsf{V}$, the essential image of j_* is a left admissible category, where j_* is the right adjoint to j^*.
- U is **admissible** if it is both left and right admissible. In this case, we get a recollement



with $\mathsf{V} \cong \mathsf{U}^\perp \cong {}^\perp\mathsf{U}$.

We now come to the related notion of semiorthogonal decompositions.

Definition 2.19. Let T be a triangulated category. A semiorthogonal decomposition on T is a sequence of strictly full triangulated subcategories $\mathsf{U}_1, \cdots, \mathsf{U}_n$ such that,

- $\mathrm{Hom}_\mathsf{T}(\mathsf{U}_i, \mathsf{U}_j) = 0$ for all $n \geq i > j \geq 1$.

- The smallest triangulated subcategory of T containing $\bigcup_{1\leq i\leq n} \mathsf{U}_i$ is T itself.

We denote a semiorthogonal decomposition by $\langle \mathsf{U}_1, \cdots, \mathsf{U}_n\rangle$.

Remark 2.20. In this work, we will only consider semiorthogonal decompositions with exactly two components. The following are easy to check,

- If $\langle \mathsf{V}, \mathsf{U}\rangle$ is a semiorthogonal decompositions, then V is left admissible and U is right admissible.
- U is right admissible if and only if $\langle \mathsf{U}^{\perp}, \mathsf{U}\rangle$ is a semiorthogonal decomposition.
- V is left admissible if and only if $\langle \mathsf{V}, {}^{\perp}\mathsf{V}\rangle$ is a semiorthogonal decomposition.

3. The abstract results

We begin by recalling the following definition from [SZ21].

Definition 3.1. [SZ21, Definition 4.1] Let S and T be triangulated categories with good metrics ([Nee20, Definition 10]) $\{\mathcal{M}_n\}$ and $\{\mathcal{N}_n\}$ respectively. Let $F : \mathsf{S} \to \mathsf{T}$ a triangulated functor. Then, we say F is a **compression functor** if for all $i > 0$, there exists a $n > 0$ such that $F(\mathcal{M}_n) \subseteq \mathcal{N}_i$.

In similar spirit, we define the following notions.

Definition 3.2. Let T be a R-linear triangulated category for a commutative ring R. Then,

(1) We say an extended good metric $\mathcal{M}$ (Definition 2.2) is a **compressed metric** if all triangulated functors $F : \mathsf{T} \to \mathsf{T}$ are compression functors with respect to $\mathcal{M}$. That is, for all $i \in \mathbb{Z}$, there exists $n \in \mathbb{Z}$ such that $F(\mathcal{M}_n) \subseteq \mathcal{M}_i$.
 We further say an extended good metric $\mathcal{M}$ is a **$\mathbb{N}$-compressed metric** if for any triangulated functor $F : \mathsf{T} \to \mathsf{T}$, there exists an integer $l \geq 0$ such that $F(\mathcal{M}_i) \subseteq \mathcal{M}_{i+l}$ for all $i \in \mathbb{Z}$.

(2) A finite generating sequence (Definition 2.3) $\mathcal{G}$ is a **compressed generating sequence** (resp. **$\mathbb{N}$-compressed generating sequence**) if the metric $\mathcal{R}^{\mathcal{G}} \cap \mathsf{T}^c$ is a compressed metric (resp. $\mathbb{N}$-compressed metric), and is equivalent (Definition 2.2) to the metric $\mathcal{M}^{\mathcal{G}}$ on T^c, see Definition 2.3 for the definition of $\mathcal{R}^{\mathcal{G}}$ and $\mathcal{M}^{\mathcal{G}}$.

(3) We say that a Serre subcategory $\mathcal{C} \subseteq \prod_{i\in\mathbb{Z}} \mathrm{Mod}(R)$ is **compressed** if it is stable under sending the indexing sequence $\{i\}_{i\in\mathbb{Z}}$ to $\{i+n\}_{i\in\mathbb{Z}}$ for any integer n. That is, if $\{M_i\}_{i\in\mathbb{Z}}$ is in $\mathcal{C}$ and $n \in \mathbb{Z}$, then $\{M_{i+n}\}_{i\in\mathbb{Z}}$ is also in $\mathcal{C}$.

The following result gives an important class of generating sequences and metrics which are compressed.

Theorem 3.3. *Let S be a triangulated category with a classical generator G, that is $\mathsf{S} = \langle G\rangle$. Then, the following finite generating filtrations are $\mathbb{N}$-compressed, see Definition 3.2(2),*

- $\mathcal{G}[n, n] := \{\Sigma^{-n}G\}$ *for all* $n \in \mathbb{Z}$.
- $\mathcal{G}'[n, n] := \{\Sigma^{n}G\}$ *for all* $n \in \mathbb{Z}$.

Proof. Let $F : \mathsf{S} \to \mathsf{S}$ be any triangulated functor. Then, $F(G) \in \mathsf{S} = \langle G\rangle$. But, $\langle G\rangle = \bigcup_{i\geq 0}\langle G\rangle^{[-i,i]}$, and so, there exists $i \geq 0$ such that $F(G) \in \langle G\rangle^{[-i,i]}$, see Definition 2.1. As

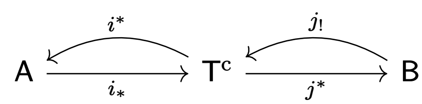
F is a triangulated functor, it preserves direct sums, summands, shifts, and extensions. So, for all $n \geq 0$, we have that

$$F(\mathcal{M}^{\mathcal{G}}_{n+i}) \subseteq \mathcal{M}^{\mathcal{G}}_{n} \text{ and } F(\mathcal{M}^{\mathcal{G}'}_{n+i}) \subseteq \mathcal{M}^{\mathcal{G}'}_{n}$$

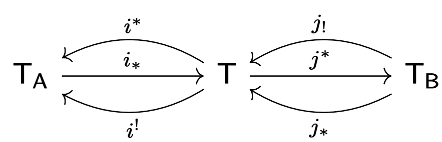
with notation as in Definition 2.3, which shows that both the metrics are $\mathbb{N}$-compressed. But, by Example 2.5, $\mathcal{R}^{\mathcal{G}} \cap \mathsf{T}^c$ is $\mathbb{N}$-equivalent (Definition 2.2) to $\mathcal{M}^{\mathcal{G}}$ and $\mathcal{R}^{\mathcal{G}'} \cap \mathsf{T}^c$ is $\mathbb{N}$-equivalent to $\mathcal{M}^{\mathcal{G}'}$ and hence these metrics are also $\mathbb{N}$-compressed, which in turn implies that the generating sequences are $\mathbb{N}$-compressed (see Definition 3.2). □

The following lemma, which is known in the literature, will help us in proving the main results of this section.

Lemma 3.4. *Let T be a compactly generated triangulated category, and suppose there exists a semiorthogonal decomposition $\langle \mathsf{A}, \mathsf{B} \rangle$ on T^c. We know this gives a colocalisation sequence on T^c as follows,*

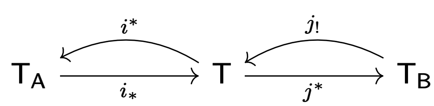


Then, this extends to a recollement on T as follows,



where $\mathsf{T}_\mathsf{A} = \mathrm{Coprod}(\mathsf{A})$ and $\mathsf{T}_\mathsf{B} = \mathrm{Coprod}(\mathsf{B})$. In particular, $\langle \mathsf{A}^\perp, \mathsf{B}^\perp \rangle$ is a semiorthogonal decomposition on T.

Proof. It is well-known, and easy to show, that the colocalisation sequence on T^c extends to one on T as follows,



We need to show that it is also a localisation sequence. It is enough to show that there exists a right adjoint to i_*. Note that T_A is a compactly generated triangulated category, and i_* preserves coproducts. So, by Neeman's adjoint functor theorem (see [Nee96, Theorem 4.1]), i_* has a right adjoint giving us the required recollement.

In particular, $\langle j_*(\mathsf{T}_\mathsf{B}), i_*(\mathsf{T}_\mathsf{A}) \rangle$ is a semiorthogonal decomposition on T. So, $j_*(\mathsf{T}_\mathsf{B}) = i_*(\mathsf{T}_\mathsf{A})^\perp = \mathrm{Coprod}(\mathsf{A})^\perp = \mathsf{A}^\perp$. As $\langle i_*(\mathsf{T}_\mathsf{A}), j_!(\mathsf{T}_\mathsf{B}) \rangle$ is also a semiorthogonal decomposition on T, we have that $i_*(\mathsf{T}_\mathsf{A}) = j_!(\mathsf{T}_\mathsf{B})^\perp = \mathrm{Coprod}(\mathsf{B})^\perp = \mathsf{B}^\perp$. And so, this shows that $\langle \mathsf{A}^\perp, \mathsf{B}^\perp \rangle$ is a semiorthogonal decomposition on T. □

Now, we get to the first main result of this section.

Theorem 3.5. *Let T a triangulated category with a generating sequence $\mathcal{G}$ (Definition 2.3), and an orthogonal metric $\mathcal{R}$ (Definition 2.2) such that for all $n \in \mathbb{Z}$, $\mathrm{Hom}_\mathsf{T}(\mathcal{G}[n,n], \mathcal{R}_i) = 0$ for all $i \gg 0$. We further assume that the metric $\mathcal{R} \cap \mathsf{T}^c$ is* *compressed, see Definition 3.2(1). Suppose we are given a localisation sequence on* T^c *as follows,*

$$\mathsf{A} \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathsf{T}^c \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathsf{B}$$

with $i_* : \mathsf{A} \to \mathsf{T}^c$ *and* $j_* : \mathsf{B} \to \mathsf{T}^c$ *inclusions of strictly full subcategories. Let* T_A *and* T_B *be the localising subcategories generated by* A *and* B *respectively inside* T*, that is,* $\mathsf{T}_\mathsf{A} = \mathrm{Coprod}(\mathsf{A})$ *and* $\mathsf{T}_\mathsf{B} = \mathrm{Coprod}(\mathsf{B})$*, see Definition 2.1. These categories have generating sequences* $\mathcal{G}_\mathsf{A} := i^!(\mathcal{G})$ *and* $\mathcal{G}_\mathsf{B} := j^*(\mathcal{G})$ *and orthogonal metrics* $\mathcal{R}^\mathsf{A} := i^!(\mathcal{R})$ *and* $\mathcal{R}^\mathsf{B} := j^*(\mathcal{R})$ *on* T_A *and* T_B *respectively.*

Then, we get a localisation sequence on the closure of the compacts $\overline{\mathsf{T}^c}$ *(see Definition 2.10) as follows,*

$$\overline{\mathsf{T}^c_\mathsf{A}} \underset{i^!}{\overset{i_*}{\rightleftarrows}} \overline{\mathsf{T}^c} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \overline{\mathsf{T}^c_\mathsf{B}}$$

with the closure of the compacts on T_A*,* T*, and* T_B *with respect to the metrics* $\mathcal{R}^\mathsf{A}$*,* $\mathcal{R}$*, and* $\mathcal{R}^\mathsf{B}$ *respectively. Further,* $\overline{\mathsf{T}^c_\mathsf{A}} = \mathsf{T}_\mathsf{A} \cap \overline{\mathsf{T}^c}$ *and* $\overline{\mathsf{T}^c_\mathsf{B}} = \mathsf{T}_\mathsf{B} \cap \overline{\mathsf{T}^c}$*.*

Proof. As noted in the proof of Lemma 3.4, we get a localisation sequence on T as follows,

$$\mathsf{T}_\mathsf{A} \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathsf{T} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathsf{T}_\mathsf{B}$$

where we denote the functors by the same symbols as the localisation sequence on T^c.

We observe now that all the four functors involved in the localisation sequence on T preserve coproducts, and hence respect homotopy colimits. The functors i_* and j^* preserve coproducts as they have right adjoints. Further, the functors $i^!$ and j_* preserve coproducts by [Nee96, Theorem 5.1] because their left adjoints preserve compact objects.

We now show that this localisation sequence restricts to the closure of the compacts. That is, for any $F \in \overline{\mathsf{T}^c}$, we want to show that $i^!(F) \in \overline{\mathsf{T}^c_\mathsf{A}}$ and $j^*(F) \in \overline{\mathsf{T}^c_\mathsf{B}}$. As $F \in \overline{\mathsf{T}^c}$, there is a sequence E_i mapping to F such that $\underrightarrow{\mathrm{Hocolim}}\, E_i \cong F$ and $\mathrm{Cone}(E_i \to E_{i+j}) \in \mathcal{R}_{i+1}$ for all $i, j \geq 0$ by Lemma 2.11. So, we get that $i^!(F) = \underrightarrow{\mathrm{Hocolim}}\, i^!(E_i)$ and $j^* = \underrightarrow{\mathrm{Hocolim}}\, j^*(E_i)$ as $i^!$ and $j^*(F)$ respect homotopy colimits. Note that $\mathrm{Cone}(i^!(E_i) \to i^!(E_{i+j})) \in \mathcal{R}^\mathsf{A}_{i+1}$ and $\mathrm{Cone}(j^*(E_i) \to j^*(E_{i+j})) \in \mathcal{R}^\mathsf{B}_{i+1}$ for all $i, j \geq 1$ by definition. Therefore, by Lemma 2.11, we get that $i^!(F) = \underrightarrow{\mathrm{Hocolim}}\, i^!(E_i) \in \overline{\mathsf{T}^c_\mathsf{A}}$ and $j^*(F) = \underrightarrow{\mathrm{Hocolim}}\, j^*(E_i) \in \overline{\mathsf{T}^c_\mathsf{B}}$, which is what we needed to show.

Finally, we need to show that $i_*(\overline{\mathsf{T}^c_\mathsf{A}}) \subseteq \overline{\mathsf{T}^c}$ and $j_*(\overline{\mathsf{T}^c_\mathsf{B}}) \subseteq \overline{\mathsf{T}^c}$. We show that $i_*(\overline{\mathsf{T}^c_\mathsf{A}}) \subseteq \overline{\mathsf{T}^c}$, and the other inclusion follows similarly. Let $A \in \overline{\mathsf{T}^c_\mathsf{A}}$. Then, there exists a sequence A_* mapping to A such that $\underrightarrow{\mathrm{Hocolim}}\, A_i \cong A$, and $\mathrm{Cone}(A_i \to A_{i+j}) \in \mathcal{R}^\mathsf{A}_{i+1} \cap \mathsf{T}^c$ for all $i, j \geq 1$. But, $\mathcal{R} \cap \mathsf{T}^c$ is assumed to be a compressed metric, and $\mathcal{R}^\mathsf{A}_i = i_* i^!(\mathcal{R}_i)$, that is, it is the image of $\mathcal{R}$ under the endofunctor $i_* i^!$. So, by the definition of compressed metric

(Definition 3.2(1)), for all $j \geq 0$, there exists a $n \geq 0$ such that $i_*i^!(\mathcal{R}_n^{\mathsf{A}} \cap \mathsf{T}^c) \subseteq \mathcal{R}_j \cap \mathsf{T}^c$, which gives us that $\underrightarrow{\text{Hocolim}}\, A_i \in \overline{\mathsf{T}^c}$ by Definition 2.10. ☐

This gives us the following corollaries.

Corollary 3.6. *Let T be a compactly generated triangulated category, with a single compact generator G such that $\text{Hom}_{\mathsf{T}}(G, \Sigma^i G) = 0$ for all $i \gg 0$. Suppose we have the localisation sequence,*

$$\mathsf{A} \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathsf{T}^c \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathsf{B}$$

with $i_* : \mathsf{A} \to \mathsf{T}^c$ and $j_* : \mathsf{B} \to \mathsf{T}^c$ inclusions of strictly full subcategories. Let T_{A} and T_{B} be the localising subcategories of T generated by A and B respectively, that is, $\mathsf{T}_{\mathsf{A}} = \text{Coprod}(\mathsf{A})$ and $\mathsf{T}_{\mathsf{B}} = \text{Coprod}(\mathsf{B})$, see Definition 2.1. Then, we get the following localisation sequence,

$$(\mathsf{T}_{\mathsf{A}})^-_c \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathsf{T}^-_c \underset{j_*}{\overset{j^*}{\rightleftarrows}} (\mathsf{T}_{\mathsf{B}})^-_c$$

with the categories T^-_c, $(\mathsf{T}_{\mathsf{A}})^-_c$, and $(\mathsf{T}_{\mathsf{B}})^-_c$ defined as in Convention 2.12 with respect to any t-structure in the preferred quasiequivalence class of t-structures on T, T_{A}, and T_{B} respectively (see Definition 2.9).

Proof. We are working with the generating sequence $\mathcal{G}$ given by $\mathcal{G}[n,n] = \{\Sigma^{-n}G\}$, which is compressed, see Theorem 3.3. Hence the metric $\mathcal{R}^{\mathcal{G}} \cap \mathsf{T}^c$ is compressed by Definition 3.2. By Convention 2.12, T^-_c is the closure of the compacts $\overline{\mathsf{T}^c}$, see Definition 2.10. Further, for any $n \in \mathbb{Z}$, we have that $\text{Hom}_{\mathsf{T}}(\mathcal{G}[n,n], \mathcal{R}_i^{\mathcal{G}}) = \text{Hom}_{\mathsf{T}}(\Sigma^{-n}G, \overline{\langle G \rangle}^{(-\infty,-i]}) = 0$ for $i \gg 0$ as G is compact, and we have $\text{Hom}_{\mathsf{T}}(G, \Sigma^i G) = 0$ for $i \gg 0$ by hypothesis. So, we get the required result from Theorem 3.5. ☐

Corollary 3.7. *Let T be a triangulated category with a single compact generator G such that $\text{Hom}_{\mathsf{T}}(G, G[-i]) = 0$ for all $i \gg 0$. Suppose we have the localisation sequence,*

$$\mathsf{A} \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathsf{T}^c \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathsf{B}$$

with $i_* : \mathsf{A} \to \mathsf{T}^c$ and $j_* : \mathsf{B} \to \mathsf{T}^c$ inclusions of strictly full subcategories. Let T_{A} and T_{B} be the localising subcategories of T generated by A and B respectively, that is, $\mathsf{T}_{\mathsf{A}} = \text{Coprod}(\mathsf{A})$ and $\mathsf{T}_{\mathsf{B}} = \text{Coprod}(\mathsf{B})$, see Definition 2.1. Then, we get the following localisation sequence

$$(\mathsf{T}_{\mathsf{A}})^+_c \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathsf{T}^+_c \underset{j_*}{\overset{j^*}{\rightleftarrows}} (\mathsf{T}_{\mathsf{B}})^+_c$$

with the categories T^+_c, $(\mathsf{T}_{\mathsf{A}})^+_c$, and $(\mathsf{T}_{\mathsf{B}})^+_c$ defined as in Convention 2.12 with respect to any co-t-structure in the preferred quasiequivalence class of co-t-structures on T, T_{A}, and T_{B} respectively (see Definition 2.9).

Proof. We are working with the compressed generating sequence $\mathcal{G}$ given by $\mathcal{G}[n,n] = \{\Sigma^n G\}$. Hence the metric $\mathcal{R}^{\mathcal{G}} \cap \mathsf{T}^c$ is compressed by Definition 3.2. By Convention 2.12, T_c^+ is the closure of the compacts $\overline{\mathsf{T}^c}$, see Definition 2.10. Further, for any $n \in \mathbb{Z}$, we have that $\operatorname{Hom}_{\mathsf{T}}(\Sigma^{-n}G, \overline{\langle G \rangle}^{[i,\infty)}) = 0$ for $i \gg 0$ as G is compact, and we have $\operatorname{Hom}_{\mathsf{T}}(G, \Sigma^i G) = 0$ for $i \gg 0$ by hypothesis. As $\mathcal{R}_n^{\mathcal{G}} = \Sigma^{-n}\mathsf{U}_G \subseteq \overline{\langle G \rangle}^{[n-1,\infty)}$ by [Bon22, Theorem 2.3.4] where $(\mathsf{U}_G, \mathsf{V}_G)$ is the co-t-structure generated by G, $\operatorname{Hom}_{\mathsf{T}}(\mathcal{G}[n,n], \mathcal{R}_i^{\mathcal{G}})$ for $i \gg 0$. So, we get the required result from Theorem 3.5. ☐

We now want to prove the corresponding results about the bounded objects in the closure of compacts T_c^b (see Definition 2.10). For that, we need a few more ingredients. We start with some notation.

Notation 3.8. Let U and S be full subcategory of T, then the full subactegory of S of objects which are right (resp. left) orthogonal to U is denoted by $\mathsf{U}_{\mathsf{S}}^{\perp}$ (resp. ${}^{\perp}\mathsf{U}_{\mathsf{S}}$). If we write $\mathsf{U}^{\perp}$ or ${}^{\perp}\mathsf{U}$ without a subscript, it is assumed that $\mathsf{S} = \mathsf{T}$.

Throughout the rest of the section, R will denote a commutative ring. We now give a definition, and then prove a result which is similar in spirit to [Bon24, Theorem 1.3.II.2].

Definition 3.9. Let T be a compactly generated R-linear triangulated category with a generating sequence $\mathcal{G}$, and a compressed abelian subcategory $\mathcal{S} \subseteq \prod_{i\in\mathbb{Z}} \operatorname{Mod}(R)$, see Definition 3.2. We define the following thick subcategory of T,

$$\mathsf{T}_{\mathcal{S},\mathcal{G}} := \Big\{ t \in \mathsf{T} : \big\{ \operatorname{Hom}_{\mathsf{T}}(h_i, t) \big\}_{i\in\mathbb{Z}} \in \mathcal{S} \text{ for all sequences } \{h_i\}_{i\in\mathbb{Z}} \text{ with } h_i \in \mathcal{M}_i^{\mathcal{G}} \Big\}$$

see Definition 2.3.

Theorem 3.10. *Let T be a compactly generated R-linear triangulated category with a $\mathbb{N}$-compressed generating sequence $\mathcal{G}$, and a compressed abelian subcategory $\mathcal{S} \subseteq \prod_{i\in\mathbb{Z}} \operatorname{Mod}(R)$, see Definition 3.2. Then, for any semiorthogonal decomposition $\langle \mathsf{A}, \mathsf{B} \rangle$ of T^c, we have that $\langle \mathsf{A}^{\perp}_{\mathsf{T}_{\mathcal{S},\mathcal{G}}}, \mathsf{B}^{\perp}_{\mathsf{T}_{\mathcal{S},\mathcal{G}}} \rangle$ is a semiorthogonal decompositions of $\mathsf{T}_{\mathcal{S},\mathcal{G}}$, see Notation 3.8 and Definition 3.9. Further, $\mathsf{B}^{\perp}_{\mathsf{T}_{\mathcal{S},\mathcal{G}}} = \operatorname{Coprod}(\mathsf{A}) \cap \mathsf{T}_{\mathcal{S},\mathcal{G}}$.*

Proof. By Lemma 3.4, we get the recollement,

$$\mathsf{T}_{\mathsf{A}} \;\underset{\overset{\xleftarrow{\;i^*\;}}{\xleftarrow[\;i^!\;]{}}}{\xrightarrow{\;i_*\;}}\; \mathsf{T} \;\underset{\overset{\xleftarrow{\;j_!\;}}{\xleftarrow[\;j_*\;]{}}}{\xrightarrow{\;j^*\;}}\; \mathsf{T}_{\mathsf{B}}$$

where $\mathsf{T}_{\mathsf{A}} = \operatorname{Coprod}(\mathsf{A})$ and $\mathsf{T}_{\mathsf{B}} = \operatorname{Coprod}(\mathsf{B})$, see Definition 2.1.

We know that $\mathsf{B}^{\perp} = j_!(\mathsf{T}_{\mathsf{B}})^{\perp} = i_*(\mathsf{T}_{\mathsf{A}})$ is a right admissible category of T, with the corresponding left admissible subcategory $\mathsf{A}^{\perp} = i_*(\mathsf{T}_{\mathsf{A}})^{\perp} = j_*(\mathsf{T}_{\mathsf{B}})$. We need to show that the corresponding localisation sequence restricts to $\mathsf{T}_{\mathcal{S},\mathcal{G}}$.

That is, we need to show that for any $t \in \mathsf{T}_{\mathcal{S},\mathcal{G}}$, we have that $i_*i^!(t), j_*j^*(t) \in \mathsf{T}_{\mathcal{S},\mathcal{G}}$. So, let $\{h_n\}_{n\in\mathbb{Z}}$ be any collection of objects such that $h_n \in \mathcal{M}_n^{\mathcal{G}}$ for all $n \in \mathbb{Z}$. As $\mathcal{G}$ is a $\mathbb{N}$-compressed generating sequence, see Definition 3.2, we get that there exists an integer $l \geq 0$ such that $i_*i^*(h_n) \in \mathcal{M}_{n+l}^{\mathcal{G}}$ for each $n \in \mathbb{Z}$.

By adjunction, we have that $\operatorname{Hom}_{\mathsf{T}}(h_n, i_*i^!(t)) = \operatorname{Hom}_{\mathsf{T}}(i_*i^*(h_n), t)$ for any $t \in \mathsf{T}$. In particular, $\{\operatorname{Hom}_{\mathsf{T}}(h_n, i_*i^!(t))\}_{n\in\mathbb{Z}} = \{\operatorname{Hom}_{\mathsf{T}}(i_*i^*(h_n), t)\}_{n\in\mathbb{Z}}$ for any $t \in \mathsf{T}_{\mathcal{S},\mathcal{G}}$. But the latter belongs to $\mathcal{S}$ by the above paragraph, the fact that $\mathcal{S}$ is compressed (see Definition 3.2(3)), and the definition of $\mathsf{T}_{\mathcal{S},\mathcal{G}}$ (Definition 3.9). And therefore we get that, $\{\mathrm{Hom}_{\mathsf{T}}(h_n, i_*i^!(t))\}_{n\in\mathbb{Z}} \in \mathcal{S}$ for any sequence $\{h_n\}_{n\in\mathbb{Z}}$ with $h_n \in \mathcal{M}^{\mathcal{G}}_n$. This shows that $i_*i^!(t) \in \mathsf{T}_{\mathcal{S},\mathcal{G}}$. Similarly, we get that $j_*j^*(t) \in \mathsf{T}_{\mathcal{S},\mathcal{G}}$, which is what we needed to show.

Finally, note that $\mathsf{B}^{\perp} = \mathrm{Coprod}(\mathsf{A})$. And so, we get that $\mathsf{B}^{\perp}_{\mathsf{T}_{\mathcal{S},\mathcal{G}}} = \mathrm{Coprod}(\mathsf{A}) \cap \mathsf{T}_{\mathcal{S},\mathcal{G}}$. □

Corollary 3.11. *Let T be a triangulated category with a $\mathbb{N}$-compressed generating sequence $\mathcal{G}$, see Definition 3.2(2). Let $\langle \mathcal{W}, \mathcal{V}\rangle$ be a semiorthogonal decomposition on T^c. Then, $\langle \mathcal{W}^{\perp}_{\mathcal{G}^{\perp}}, \mathcal{V}^{\perp}_{\mathcal{G}^{\perp}}\rangle$ is a semiorthogonal decomposition on $\mathcal{G}^{\perp}$, see Notation 3.8. Further, $\mathcal{V}^{\perp}_{\mathcal{G}^{\perp}} = \mathrm{Coprod}(\mathcal{W}) \cap \mathcal{G}^{\perp}$, see Definition 2.1.*

Proof. This follows from Theorem 3.10 by taking $R = \mathbb{Z}$ and $\mathcal{S} \subseteq \prod_{i\in\mathbb{Z}} \mathrm{Mod}(\mathbb{Z})$ to be all the sequences of abelian groups $\{M_i\}_{i\in\mathbb{Z}}$ such that $M_i = 0$ for all $i \gg 0$. We just need to show that $\mathsf{T}_{\mathcal{S},\mathcal{G}} = \mathcal{G}^{\perp}$, see (Definition 3.9).

As $\mathcal{G}$ is a $\mathbb{N}$-compressed generating sequence, in particular it is a finite generating sequence, see Definition 3.2. So, $\mathcal{G}[n, n] = \{G_{n,i}\}_{i=1}^{j_n}$ for some objects $G_{n,i} \in \mathsf{T}^c$ and $j_n \geq 0$ for all $n \in \mathbb{Z}$. Consider the sequence $\{h_n\} \in \mathcal{S}$ with $h_n = \oplus_{i=1}^{j-n} G_{-n,i}$ for all $n \in \mathbb{Z}$. Note that $h_n \in \mathcal{M}^{\mathcal{G}}_n$ (Definition 2.3) for all $n \in \mathbb{Z}$. If $t \in \mathsf{T}_{\mathcal{S},\mathcal{G}}$, $\mathrm{Hom}_{\mathsf{T}}(h_n, t) = 0$ for all $n \gg 0$ by Definition 3.9, which immediately implies that $t \in \mathcal{G}^{\perp}$, which shows that $\mathsf{T}_{\mathcal{S},\mathcal{G}} \subseteq \mathcal{G}^{\perp}$. Conversely, let $t \in \mathcal{G}^{\perp}$. So, $\mathrm{Hom}_{\mathsf{T}}(t, \mathcal{M}^{\mathcal{G}}_n) = 0$ for $n \gg 0$, and hence $t \in \mathsf{T}_{\mathcal{S},\mathcal{G}}$. □

As an immediate corollary, we get the following result.

Corollary 3.12. *Let T be triangulated category with a single compact generator G. Let $\langle \mathcal{W}, \mathcal{V}\rangle$ be a semiorthoginal decomposition on T^c. Then,*

- *$\langle \mathcal{W}^{\perp}_{\mathsf{T}^+}, \mathcal{V}^{\perp}_{\mathsf{T}^+}\rangle$ is a semiorthogonal decomposition on $\mathsf{T}^+ := \bigcup_{n\in\mathbb{Z}} \Sigma^n \mathsf{V}$, where (U, V) is a t-structure in the preferred equivalence class of t-structures on T, see Definition 2.9.*
- *$\langle \mathcal{W}^{\perp}_{\mathsf{T}^-}, \mathcal{V}^{\perp}_{\mathsf{T}^-}\rangle$ restricts to a semiorthogonal decomposition on $\mathsf{T}^- := \bigcup_{n\in\mathbb{Z}} \Sigma^n \mathsf{V}$, where (U, V) is a co-t-structure in the preferred equivalence class of co-t-structures on T, see Definition 2.9.*

Proof. We only prove (1), as (2) follows exactly the same way. Let G be a compact generator for T, which exists by assumption. Let $\mathcal{G}$ be the finite generating filtration (Definition 2.3) given by $\mathcal{G}[n, n] = \{\Sigma^{-n} G\}$. By Theorem 3.3, $\mathcal{G}$ is $\mathbb{N}$-compressed. Further, the category $\mathsf{T}^+ = \mathcal{G}^{\perp} = \bigcup_{n\in\mathbb{Z}} (\mathcal{M}^{\mathcal{G}}_n)^{\perp}$, see Definition 2.3 for the definition of $\mathcal{M}^{\mathcal{G}}$. So, (1) follows from Corollary 3.11. □

Finally, combining the results, we get the following result for the bounded objects in the closure of the compacts.

Theorem 3.13. *Let T be a triangulated category with a $\mathbb{N}$-compressed generating sequence $\mathcal{G}$ (Definition 3.2), and an orthogonal good metric $\mathcal{R}$ (Definition 2.2) such that for all $n \in \mathbb{Z}$, $\mathrm{Hom}_{\mathsf{T}}(\mathcal{G}[n, n], \mathcal{R}_i) = 0$ for all $i \gg 0$. Then, for any admissible category A of T^c, we have that $\mathrm{Coprod}(\mathsf{A}) \cap \mathsf{T}^b_c$ is a right admissible subcategory of T^b_c (see Definitions 2.1 and 2.10).*

Proof. As A is admissible, we get that $\mathsf{B} = {}^{\perp}\mathsf{A}_{\mathsf{T}^c}$ is right admissible, see Notation 3.8. So, by Corollary 3.11, $\mathsf{B}^{\perp}_{\mathcal{G}^{\perp}} = \mathrm{Coprod}(\mathsf{A}) \cap \mathcal{G}^{\perp}$ is a right admissible subcategory of $\mathcal{G}^{\perp}$. As A is admissible, we have that $\mathrm{Coprod}(\mathsf{A}) \cap \overline{\mathsf{T}^c}$ is also admissible by Theorem 3.5. As these are compatible, we get that $\mathrm{Coprod}(\mathsf{A}) \cap \overline{\mathsf{T}^c} \cap \mathcal{G}^{\perp} = \mathrm{Coprod}(\mathsf{A}) \cap \mathsf{T}^b_c$ is a right admissible subcategory of $\mathsf{T}^b_c = \overline{\mathsf{T}^c} \cap \mathcal{G}^{\perp}$. □

As immediate corollaries, we get the following results.

Corollary 3.14. *Let T be a triangulated category with a single compact generator G such that $\mathrm{Hom}_{\mathsf{T}}(G, \Sigma^n G) = 0$ for all $n \gg 0$. We equip it with the $\mathbb{N}$-compressed generating sequence (Definition 3.2) $\mathcal{G}$ given by $\mathcal{G}[n, n] = \{\Sigma^{-n} G\}$ for all $n \in \mathbb{Z}$, see Theorem 3.3. Then, for any admissible category A of T^c, we have that $\mathrm{Coprod}(\mathsf{A}) \cap \mathsf{T}^b_c$ (see Definition 2.1) is a right admissible subcategory of T^b_c (Definition 2.10).*

Proof. This is immediate from Theorem 3.13. □

Corollary 3.15. *Let T be a triangulated category with a single compact generator G such that $\mathrm{Hom}_{\mathsf{T}}(G, \Sigma^n G) = 0$ for all $n << 0$. We equip it with the $\mathbb{N}$-compressed generating sequence (Definition 3.2) $\mathcal{G}$ given by $\mathcal{G}[n, n] = \{\Sigma^{n} G\}$ for all $n \in \mathbb{Z}$, see Theorem 3.3. Then, for any admissible category A of T^c, we have that $\mathrm{Coprod}(\mathsf{A}) \cap \mathsf{T}^b_c$ (see Definition 2.1) is a right admissible subcategory of T^b_c (Definition 2.10).*

Proof. This is immediate from Theorem 3.13. □

4. Examples of co-approximability

In this section we will give some examples of weakly co-approximable and weakly co-quasiapproximable triangulated categories coming from algebraic geometry. The main source of examples for (weak) co-approximability is the homotopy category of injectives for a nice enough Noetherian scheme, algebra, or stack. As mentioned in the introduction, this will help us compute the closure of the compacts for the homotopy category of injectives, which we will do in §5.

We begin in a more general setting. Recall that a locally Noetherian Grothendieck abelian category is a Grothendieck abelian category $\mathcal{C}$, containing a set of Noetherian objects $\mathcal{D}$ which generate it, that is, every object is a quotient of coproduct of objects in $\mathcal{D}$. We will denote the full subcategory of Noetherian objects by $\mathrm{noeth}\,\mathcal{C}$. The category we will be interested in is the homotopy category of injectives $\mathbf{K}(\mathrm{Inj}\,\mathcal{C})$. By [Kra05, Proposition 2.3], $\mathbf{K}(\mathrm{Inj}\,\mathcal{C})$ is compactly generated, and the full subcategory of compacts is equivalent to the bounded derived category of Noetherian objects $\mathbf{D}^b(\mathrm{noeth}\,\mathcal{C})$.

Notation 4.1. Let $\mathcal{C}$ be a locally Noetherian Grothendieck abelian category. Then,

- $(\mathbf{D}(\mathcal{C})^{\leq 0}, \mathbf{D}(\mathcal{C})^{\geq 0})$ denotes the standard t-structure on $\mathbf{D}(\mathcal{C})$, where the truncation triangles are given by the canonical truncation of complexes.
- $(\mathbf{K}(\mathrm{Inj}\,\mathcal{C})^{\geq 0}, \mathbf{K}(\mathrm{Inj}\,\mathcal{C})^{\leq 0})$ denotes the standard co-t-structure on $\mathbf{K}(\mathrm{Inj}\,\mathcal{C})$, where the truncation triangles are given by the brutal truncation of complexes.

For a Noetherian scheme X, we will work with the locally Noetherian Grothendieck abelian category of quasicoherent sheaves $\mathrm{Qcoh}\,X$.

We start with a couple of easy and well known lemmas.

Lemma 4.2. *Let $\mathcal{C}$ be a locally Noetherian Grothendieck abelian category and we denote by $p : \mathbf{K}(\operatorname{Inj}\mathcal{C}) \to \mathbf{D}(\mathcal{C})$ the Verdier quotient map, see* [Kra05, Proposition 3.6]*. Then, this restricts to an equivalence $p : \mathbf{K}(\operatorname{Inj}\mathcal{C})^{\geq 0} \to \mathbf{D}(\mathcal{C})^{\geq 0}$.*

Proof. This follows trivially as $\operatorname{Hom}_{\mathbf{K}(\operatorname{Inj}\mathcal{C})}(E,F) \cong \operatorname{Hom}_{\mathbf{D}(\mathcal{C})}(E,F)$ if F is a bounded below complex of injective objects, and E is an arbitrary complex of injective objects, see [Sta24, Tag 05TG]. □

Remark 4.3. In particular Lemma 4.2 implies that for any $G \in \mathbf{D}^{\mathrm{b}}(\operatorname{noeth}\mathcal{C})$, the subcategories $\overline{\langle G\rangle}_C^{[A,B]}$ (see Definition 2.1) for any integers A, B and positive integer C remain the same viewed in either $\mathbf{K}(\operatorname{Inj}\mathcal{C})$ or $\mathbf{D}(\mathcal{C})$, that is they are equivalent under the functor p. We will confuse these equivalent categories freely throughout the rest of this document.

Lemma 4.4. [Kra05, Lemma 2.1] *Let $\mathcal{C}$ be a locally Noetherian Grothendieck abelian category. Let F be a bounded below complex, and I an arbitrary complex of injective sheaves. Then, $\operatorname{Hom}_{\mathbf{K}(\mathcal{C})}(F,I) \cong \operatorname{Hom}_{\mathbf{K}(\operatorname{Inj}\mathcal{C})}(I_F,I)$, where I_F is an injective resolution of F.*

Proof. We complete the map that we get from the injective resolution to a triangle $F \to I_F \to \operatorname{Cone}(F \to I_F) \to \Sigma F$ in $\mathbf{K}(\mathcal{C})$. Note that $\operatorname{Cone}(F \to I_F)$ is an acyclic complex. So, $\operatorname{Hom}(\operatorname{Cone}(F \to I_F), \sigma^{\geq n} I) = 0$ by [Sta24, Tag 013R] for any $n \in \mathbb{Z}$, where $\sigma^{\geq n} I$ denote the brutal truncation of I. But, as F is a bounded below complex, there exists an integer N such that $F^i = 0$ for all $i \leq N$. So,

$$\operatorname{Hom}(\Sigma^n \operatorname{Cone}(F \to I_F), I) = \operatorname{Hom}(\operatorname{Cone}(F \to I_F), \sigma^{\geq N-2} I) = 0$$

And so, $\operatorname{Hom}_{\mathbf{K}(\mathcal{C})}(F,I) \cong \operatorname{Hom}_{\mathbf{K}(\operatorname{Inj}\mathcal{C})}(I_F,I)$ □

We now come to a result on weak co-approximability.

Proposition 4.5. *Let $\mathcal{C}$ be a locally Noetherian Grothendieck abelian category. Suppose there exists an object $\hat{G} \in \mathbf{D}^{\mathrm{b}}(\operatorname{noeth}(\mathcal{C}))$ and a positive integer N such that,*

$$\mathbf{D}(\mathcal{C})^{\geq 0} = \overline{\langle \hat{G}\rangle}^{[-N,N]} \star \mathbf{D}(\mathcal{C})^{\geq 1}$$

see Definition 2.1. Then,

(i) $\hat{G}$ is a compact generator for $\mathbf{K}(\operatorname{Inj}\mathcal{C})$.
(ii) $\mathbf{K}(\operatorname{Inj}\mathcal{C})$ is weakly co-approximable.
(iii) The co-t-structure $(\mathbf{K}(\operatorname{Inj}\mathcal{C})^{\geq 0}, \mathbf{K}(\operatorname{Inj}\mathcal{C})^{\leq 0})$ lies in the preferred quasiequivalence class (see Definition 2.9).

Proof. We first show that $\hat{G}$ is a compact generator for $\mathbf{K}(\operatorname{Inj}\mathcal{C})$. Let $F \in \mathbf{K}(\operatorname{Inj}\mathcal{C})^{\geq 0} \cong \mathbf{D}(\mathcal{C})^{\geq 0}$, (see Lemma 4.2). Then, by induction we will construct a sequence $F_1 \to F_2 \to F_3 \to F_4 \to \cdots$ mapping to F such that for all $i \geq 1$,

- $F_i \in \overline{\langle \hat{G}\rangle}^{[-N,\infty)}$ for some integer $N \geq 0$ independent of i.
- $D_n := \operatorname{Cone}(F_i \to F) \in \mathbf{K}(\operatorname{Inj}\mathcal{C})^{\geq i} \cong \mathbf{D}(\mathcal{C})^{\geq i}$.

We get F_1 from our assumption that $\mathbf{D}(\mathcal{C})^{\geq 0} = \overline{\langle \hat{G}\rangle}^{[-N,N]} \star \mathbf{D}(\mathcal{C})^{\geq 1}$. Now, suppose we have the sequence $F_1 \to \cdots \to F_n$. By the induction hypothesis, we know that $D_n := \operatorname{Cone}(F_n \to F) \in \mathbf{K}(\operatorname{Inj}\mathcal{C})^{\geq n} \cong \mathbf{D}(\mathcal{C})^{\geq n}$. So, $\Sigma^n D_n \in \mathbf{K}(\operatorname{Inj}\mathcal{C})^{\geq 0} \cong \mathbf{D}(\mathcal{C})^{\geq 0}$, and hence again by our assumption, we get a triangle $\widetilde{F} \to D_n \to D_{n+1} \to \Sigma\widetilde{F}$ such that $\widetilde{F} \in \overline{\langle \hat{G} \rangle}^{[-N+n,N+n]}$ and $D_{n+1} \in \mathbf{K}(\operatorname{Inj} \mathcal{C})^{\geq n+1}$. Applying the octahedral axiom to $F \to D_n \to D_{n+1}$, we get,

$$\begin{array}{ccccc} F_n & \longrightarrow & F_{n+1} & \longrightarrow & \widetilde{F} \\ \downarrow & & \downarrow & & \downarrow \\ F_n & \longrightarrow & F & \longrightarrow & D_n \\ & & \downarrow & & \downarrow \\ & & D_{n+1} & \longrightarrow & D_{n+1} \end{array}$$

The middle column gives us the required triangle, as,

$$F_{n+1} \in \overline{\langle \hat{G} \rangle}^{[-N,\infty)} \star \overline{\langle \hat{G} \rangle}^{[-N+n,N+n]} \subseteq \overline{\langle \hat{G} \rangle}^{[-N,\infty)}$$

see Definition 2.1. This gives us the required sequence mapping to F by induction.

Consider the non-canonical map $\underrightarrow{\operatorname{Hocolim}} F_i \to F$. For any object $H \in \mathbf{D}^{\mathrm{b}}(\operatorname{noeth} \mathcal{C})$, we have the map given by the composite

$$\underrightarrow{\operatorname{colim}} \operatorname{Hom}(H, F_i) \xrightarrow{\sim} \operatorname{Hom}(H, \underrightarrow{\operatorname{Hocolim}} F_i) \to \operatorname{Hom}(H, F)$$

where the first map is an isomorphism by [Nee96, Lemma 2.8]. In the triangle $\Sigma^{-1}D_i \to F_i \to F \to D_i$, with $D_i \in \mathbf{K}(\operatorname{Inj} \mathcal{C})^{\geq i}$ for all $i \geq 1$, we get that,

$$\operatorname{Hom}(H, \Sigma^{-1}D_i) = \operatorname{Hom}(H, D_i) = 0 \text{ for all } i \gg 0$$

by Lemma 4.4. So, by applying the functor $\operatorname{Hom}(H, -)$ to the triangle, we get that $\operatorname{Hom}(H, F_i) \cong \operatorname{Hom}(H, F)$ for $i \gg 0$. And so, we get that $\underrightarrow{\operatorname{colim}} \operatorname{Hom}(H, F_i) \cong \operatorname{Hom}(H, F)$ for all $H \in \mathbf{D}^{\mathrm{b}}(\operatorname{noeth} \mathcal{C})$. Hence, as $\mathbf{D}^{\mathrm{b}}(\operatorname{noeth} \mathcal{C})$ compactly generates $\mathbf{K}(\operatorname{Inj} \mathcal{C})$, by [Nee96, Lemma 2.8] we get that $F \cong \underrightarrow{\operatorname{Hocolim}} F_i \in \overline{\langle \hat{G} \rangle}^{[-N-1,\infty)}$. As F was an arbitrary object of $\mathbf{K}(\operatorname{Inj} \mathcal{C})^{\geq 0}$, we have that $\mathbf{K}(\operatorname{Inj} \mathcal{C})^{\geq 0} \subseteq \overline{\langle \hat{G} \rangle}^{[-N-1,\infty)}$. And therefore, $\bigcup_{n\in\mathbb{Z}} \mathbf{K}(\operatorname{Inj} \mathcal{C})^{\geq n} \subseteq \overline{\langle \hat{G} \rangle}$.

Now, let $F \in \mathbf{K}(\operatorname{Inj} \mathcal{C})$. Let us choose a representative complex of injectives for it, which we will also denote F. Let its brutal truncations be denoted by $\sigma^{\geq -i}F$, where we have $\sigma^{\geq -i}F \in \mathbf{K}(\operatorname{Inj} \mathcal{C})^{\geq -i}$. Then, it is easy to see that $\underrightarrow{\operatorname{Hocolim}}\, \sigma^{\geq -i}F = F$, which proves that $\mathbf{K}(\operatorname{Inj} \mathcal{C}) = \overline{\langle \hat{G} \rangle}$, which proves (i).

Now we prove (ii). For any object $H \in \mathbf{D}^{\mathrm{b}}(\operatorname{noeth} \mathcal{C})$, there exists some $N_H > 0$ such that $\Sigma^{-N_H}H \in \mathbf{K}(\operatorname{Inj} \mathcal{C})^{\geq 0}$ and $\operatorname{Hom}_{\mathsf{T}}(\Sigma^i H, \mathbf{K}(\operatorname{Inj} \mathcal{C})^{\geq 0}) = 0$ for all $i \geq N_H$. In particular, this is true for $\hat{G} \in \mathbf{D}^{\mathrm{b}}(\operatorname{noeth} \mathcal{C})$. Now, let $F \in \mathbf{K}(\operatorname{Inj} \mathcal{C})^{\geq 0} \cong \mathbf{D}(\mathcal{C})^{\geq 0}$. Then, by our assumption, there exists a triangle $E \to F \to D \to \Sigma E$ with $E \in \overline{\langle \hat{G} \rangle}^{[N,N]}$ and $D \in \mathbf{D}(\mathcal{C})^{\geq 1} \cong \mathbf{K}(\operatorname{Inj} \mathcal{C})^{\geq 1}$. So, we have proven the weak co-approximability of $\mathbf{K}(\operatorname{Inj} \mathcal{C})$ with the standard co-t-structure, and the integer $\max(N_{\hat{G}}, N)$.

Finally, (iii) immediately follows from Lemma 2.15 from the proof of (ii). □

We now move to the setting of algebraic geometry. We begin by recalling certain notions related to the regular locus of a scheme to state some of the later results.

Definition 4.6. Let X be a Noetherian scheme. Recall that the regular locus $\operatorname{reg} X$ is the collection of points $p \in X$ such that the stalk $\mathcal{O}_{X,p}$ is a regular local ring. Then,

- X is J-0 if there is a non-empty open subset contained in $\operatorname{reg} X$.
- X is J-1 if $\operatorname{reg} X$ is open. Note that this does not imply in general that X is J-0 as $\operatorname{reg} X$ can be empty. But, if X is reduced, then it being J-1 implies being J-0 as the regular locus of a Noetherian reduced scheme is non-empty. The non-emptiness of the regular locus follows from the fact that for a reduced scheme, the local rings at the generic points of each irreducible component is regular.
- X is J-2 if for every morphism $f : Y \to X$ of finite type, the regular locus $\operatorname{reg} Y$ is open in Y.

We now prove the weak approximability of $\mathbf{K}(\operatorname{Inj} X)$ under a mild hypothesis. For that, the main theorem we need to prove in preparation is the following. We state it here, and then prove it using a sequence of lemmas.

Theorem 4.7. *Let X be a finite dimensional Noetherian scheme such that each integral closed subscheme is J-0 and let T be any triangulated subcategory of $\mathbf{D}(\operatorname{Qcoh} X)$ which is closed under the truncations corresponding to the standard t-structure, that is, the canonical truncations. Then, there is an object $\hat{G}$ in $\mathbf{D}^{\mathrm{b}}(\operatorname{coh} X)$ such that,*

(1) There exists an integer $N \geq 0$ with $\mathsf{T} \cap \operatorname{Qcoh} X \subseteq \overline{\langle \hat{G} \rangle}^{[-N,N]}$.
(2) There exist integers $N_{p,q} \geq 0$ for each pair of integers $p \leq q$ such that

$$\mathsf{T} \cap \mathbf{D}(\operatorname{Qcoh} X)^{\geq p} \cap \mathbf{D}(\operatorname{Qcoh} X)^{\leq q} \subseteq \overline{\langle \hat{G} \rangle}^{[-N_{p,q},N_{p,q}]}$$

see Definition 2.1.

Remark 4.8. It is clear that it is enough to show Theorem 4.7 for $\mathsf{T} = \mathbf{D}(\operatorname{Qcoh} X)$. In fact, we are only interested in the statement for $\mathsf{T} = \mathbf{D}(\operatorname{Qcoh} X)$, and we introduce T just as a placeholder as it makes it easier to state and use the aforementioned sequence of lemmas.

Further, note that Theorem 4.7(1) $\implies$ (2). This can be shown by an induction on $(q-p)$ as follows. Suppose $F \in \mathsf{T} \cap \mathbf{D}(\operatorname{Qcoh} X)^{\geq p} \cap \mathbf{D}(\operatorname{Qcoh} X)^{\leq q}$ for integers p and q such that $q - p \geq 1$. Then, we have the triangle $\tau^{\leq q-1}F \to F \to \tau^{\geq q}F \to \Sigma\tau^{\leq q-1}F$ with $\tau^{\leq q-1}F \in \mathsf{T} \cap \mathbf{D}(\operatorname{Qcoh} X)^{\geq p} \cap \mathbf{D}(\operatorname{Qcoh} X)^{\leq q-1}$ and $\tau^{\geq q}F \in \mathsf{T} \cap \mathbf{D}(\operatorname{Qcoh} X)^{\geq q} \cap \mathbf{D}(\operatorname{Qcoh} X)^{\leq q}$ given by the canonical truncations. But,

$$\mathsf{T} \cap \mathbf{D}(\operatorname{Qcoh} X)^{\geq q} \cap \mathbf{D}(\operatorname{Qcoh} X)^{\leq q} = \Sigma^{-q}(\mathsf{T} \cap \mathbf{D}(\operatorname{Qcoh} X)^{\geq 0} \cap \mathbf{D}(\operatorname{Qcoh} X)^{\leq 0})$$

so this shows the inductive step.

Lemma 4.9. *Let X be Noetherian scheme and $i : Z \to X$ a closed immersion such that the conclusion of Theorem 4.7 holds for the scheme Z. Then, the conclusion of Theorem 4.7 holds for X with $\mathsf{T} = \mathbf{D}_Z(\operatorname{Qcoh} X)$, which is the thick subcategory of $\mathbf{D}(\operatorname{Qcoh} X)$ consisting of complexes with cohomology supported on Z.*

Proof. By Remark 4.8, it is enough to show that Theorem 4.7(1) holds for X with $\mathsf{T} = \mathbf{D}_Z(\operatorname{Qcoh} X)$. Let $\mathcal{I}$ be the sheaf of ideals corresponding to the closed immersion

$i : Z \to X$. Let $F \in \mathrm{Qcoh}_Z(X) = \mathbf{D}_Z(\mathrm{Qcoh}\, X) \cap \mathrm{Qcoh}\, X$. For each $n \geq 0$, let F_n denote the subsheaf of F which is annihilated by $\mathcal{I}^n$. That is, we define the sections of F_n for each open subset $U \subseteq X$ by, $F_n(U) := \{s \in F(U) : \mathcal{I}(U)^n \cdot s = 0\}$. Then, we get an exhaustive filtration of F as follows,

$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots \subseteq F$$

So, $F = \varinjlim F_n$ in $\mathrm{Qcoh}\, X$, which gives us that in the derived category $F \cong \underrightarrow{\mathrm{Hocolim}}\, F_n$ by [BN93, Remark 2.2].

Note that each successive quotient, F_{n+1}/F_n is annihilated by $\mathcal{I}$, and so is an $\mathcal{O}_X/\mathcal{I}$-module, that is, $F_{i+1}/F_i \in i_*(\mathrm{Qcoh}\, Z)$. Now, from the short exact sequences $0 \to F_n \to F_{n+1} \to F_{n+1}/F_n \to 0$, we get triangles $F_n \to F_{n+1} \to F_{n+1}/F_n \to \Sigma F_n$ for each $n \geq 0$. This gives us that $F_{n+1} \in F_n \star (F_{n+1}/F_n)$ for each $n \geq 0$.

As $F \cong \underrightarrow{\mathrm{Hocolim}}\, F_n$, we get that $F \in \overline{\langle \{F_{n+1}/F_n\}_{n \geq 0} \rangle}^{[-1,1]}$, see Definition 2.1. But, note that $\overline{\langle \{F_{n+1}/F_n\}_{n \geq 0} \rangle}^{[-1,1]} \subseteq \overline{\langle i_* \mathbf{D}(\mathrm{Qcoh}\, Z) \rangle}^{[-1,1]}$. By the hypothesis there exists an object $\widetilde{G} \in \mathbf{D}^{\mathrm{b}}(\mathrm{coh}(Z))$ and $N \geq 1$ such that, $\mathrm{Qcoh}\, Z \subseteq \overline{\langle \widetilde{G} \rangle}^{[-N,N]}$. Define $\hat{G} := i_*(\widetilde{G})$. Then,

$$\mathrm{Qcoh}_Z(X) \subseteq \overline{\langle i_*(\mathrm{Qcoh}\, Z) \rangle}^{[-1,1]} \subseteq i_*\Big(\overline{\langle \widetilde{G} \rangle}^{[-N-1,N+1]}\Big) \subseteq \overline{\langle \hat{G} \rangle}^{[-N-1,N+1]}$$

which gives us the required result. □

Lemma 4.10. *Let X be a Noetherian scheme with closed subschemes Z and Z' such that $X = Z \cup Z'$ topologically. If the conclusion of Theorem 4.7 holds for Z and Z', then it holds for X.*

Proof. First of all, we immediately get that the result holds for X with $\mathsf{T} = \mathbf{D}_Z(\mathrm{Qcoh}\, X)$ and $\mathsf{T} = \mathbf{D}_{Z'}(\mathrm{Qcoh}\, X)$ by Lemma 4.9. By Remark 4.8, it is enough to show that Theorem 4.7(1) holds for X with $\mathsf{T} = \mathbf{D}(\mathrm{Qcoh}\, X)$. Let $U = X \setminus Z$ and $U' = X \setminus Z'$ with corresponding open immersions $j : U \to X$ and $j' : U' \to X$.

Let $F \in \mathrm{Qcoh}\, X$. Consider the triangle $F' \to F \to \mathbb{R}j_* j^* F \to \Sigma F'$, where the second map is the unit of adjunction. Note that when restricted to U, we get that the unit of adjunction is an isomorphism, and so $j^*(F') \cong 0$. And so, $F' \in \mathbf{D}_Z(\mathrm{Qcoh}\, X)$. By [Lip09, Proposition 3.9.2], $\mathbb{R}j_* j^* F \in \mathbf{D}(\mathrm{Qcoh}\, X)^{\geq 0} \cap \mathbf{D}(\mathrm{Qcoh}\, X)^{\leq t-1}$ for some positive integer t. So, as $F' \in (\Sigma^{-1} \mathbb{R}j_* j^* F) \star F$, we get that,

$$F' \in \mathbf{D}_Z(\mathrm{Qcoh}\, X)^{\geq 0} \cap \mathbf{D}_Z(\mathrm{Qcoh}\, X)^{\leq t}$$

This in turn implies that,

$$F \in \big(\mathbf{D}_Z(\mathrm{Qcoh}\, X)^{\geq 0} \cap \mathbf{D}_Z(\mathrm{Qcoh}\, X)^{\leq t}\big) \star \mathbb{R}j_*(\mathrm{Qcoh}\, U)$$

for some positive integer t.

Further, note that $j'^{,*} \mathbb{R}j_* \mathbf{D}(\mathrm{Qcoh}\, U) = 0$ from a simple application of flat base change. This immediately implies that $\mathbb{R}j_*\, \mathrm{Qcoh}\, U \subseteq \mathbf{D}_{Z'}(\mathrm{Qcoh}\, X)^{\geq 0} \cap \mathbf{D}_{Z'}(\mathrm{Qcoh}\, X)^{\leq t-1}$. So, we get that,

$$\mathrm{Qcoh}\, X = \big(\mathbf{D}_Z(\mathrm{Qcoh}\, X)^{\geq 0} \cap \mathbf{D}_Z(\mathrm{Qcoh}\, X)^{\leq t}\big) \star \big(\mathbf{D}_{Z'}(\mathrm{Qcoh}\, X)^{\geq 0} \cap \mathbf{D}_{Z'}(\mathrm{Qcoh}\, X)^{\leq t-1}\big)$$

which immediately gives us the required result as the conclusion of Theorem 4.7 holds for X with $\mathsf{T} = \mathbf{D}_Z(\mathrm{Qcoh}\, X)$ and $\mathsf{T} = \mathbf{D}_{Z'}(\mathrm{Qcoh}\, X)$. □

Now we can prove Theorem 4.7.

Proof of Theorem 4.7. Let X be a Noetherian scheme such that each integral closed subscheme is J-0. We start by observing some reductions we can do. First of all, by Remark 4.8, it is enough to prove Theorem 4.7(1) for $\mathsf{T} = \mathbf{D}(\operatorname{Qcoh} X)$. Further, if the result holds for all the irreducible components of X, then it holds for X using Lemma 4.10. Finally, let X_{red} be the reduced scheme corresponding to X with the corresponding closed immersion $i : X_{\text{red}} \to X$. If we know that the conclusion of Theorem 4.7 holds for X_{red}, then it also holds for X by Lemma 4.9. So we can assume that the scheme is reduced and irreducible, that is, the scheme is integral.

We now proceed by induction on the dimension of the scheme X. For the base case, let X be a integral scheme such that $\dim(X) = 0$. Then X is just $\operatorname{Spec}(k)$ for a field k. Suppose $F \in \operatorname{Qcoh} X \cong \operatorname{Mod} k$. As F is a vector space over k, $F \in \overline{\langle k \rangle}^{[0,0]}$ (Definition 2.1), and hence we get the required result in this case with $\hat{G} := k$.

Now, we assume the result is known for dimension smaller than $\dim(X)$. As the scheme is integral, there exists a non-empty affine open set $U = \operatorname{Spec}(R) \subseteq \operatorname{reg} X$ by hypothesis. Let the corresponding open immersion be $j : U \to X$. Let $F \in \operatorname{Qcoh} X$. Then, the restriction to the open set U, $j^*F \in \operatorname{Qcoh} U$. As $U = \operatorname{Spec}(R) \subseteq \operatorname{reg} X$, the ring R is regular. So R is a regular ring of Krull dimensional less than or equal to $\dim(X) = n$, and hence the global dimension is less than or equal to n. So, j^*F must have a projective resolution of length less than or equal to n. This gives us that $j^*F \in \overline{\langle R \rangle}^{[-n,n]} = \overline{\langle j^*\mathcal{O}_X \rangle}^{[-n,n]}$.

Consider the triangle $F' \to F \to \mathbb{R}j_*j^*F \to \Sigma F'$, where the second map is the unit of adjunction. Note that when restricted to U, we get that the unit of adjunction is an isomorphism, and so $j^*(F') \cong 0$. Let $Z = X - U$. Then, $F' \in \mathbf{D}_Z(\operatorname{Qcoh} X)$. By [Lip09, Proposition 3.9.2], there exists $t \geq 0$ such that

$$\mathbb{R}j_*j^*F \in \mathbf{D}(\operatorname{Qcoh} X)^{\geq 0} \cap \mathbf{D}(\operatorname{Qcoh} X)^{\leq t-1}$$

So, as $F' \in (\Sigma^{-1}\mathbb{R}j_*j^*F) \star F$ we get that,

$$F' \in \mathbf{D}_Z(\operatorname{Qcoh} X)^{\geq 0} \cap \mathbf{D}_Z(\operatorname{Qcoh} X)^{\leq t}$$

This gives us that,

$$F \in (\mathbf{D}_Z(\operatorname{Qcoh} X)^{\geq 0} \cap \mathbf{D}_Z(\operatorname{Qcoh} X)^{\leq t}) \star \overline{\langle \mathbb{R}j_*j^*\mathcal{O}_X \rangle}^{[-n,n]}$$

as $F \in F' \star \mathbb{R}j_*j^*F$, and as $\mathbb{R}j_*j^*F \in \mathbb{R}j_*(\overline{\langle j^*\mathcal{O}_X \rangle}^{[-n,n]}) \subseteq \overline{\langle \mathbb{R}j_*j^*\mathcal{O}_X \rangle}^{[-n,n]}$ by the previous paragraph. Note that the integer t can be chosen independent of the choice of F in $\operatorname{Qcoh} X$.

Consider the triangle $Q \to \mathcal{O}_X \to \mathbb{R}j_*j^*\mathcal{O}_X \to \Sigma Q$ in $\mathbf{D}(\operatorname{Qcoh} X)$ coming from the unit of adjunction. Note that when restricted to U, we get that the unit of adjunction is an isomorphism, and so $j^*(Q) \cong 0$. Therefore, $\Sigma Q \in \mathbf{D}_Z(\operatorname{Qcoh} X)^{\geq -1} \cap \mathbf{D}_Z(\operatorname{Qcoh} X)^{\leq t}$. Further, $\mathcal{O}_X \in \langle G \rangle^{[-C,C]}$ for a compact generator $G \in \mathbf{D}^{\text{perf}}(X)$ and some positive integer C. So,

$$\mathbb{R}j_*j^*\mathcal{O}_X \in \langle G \rangle^{[-C,C]} \star (\mathbf{D}_Z(\operatorname{Qcoh} X)^{\geq -1} \cap \mathbf{D}_Z(\operatorname{Qcoh} X)^{\leq t})$$

Now, let $i : Z \to X$ be the closed immersion where Z is given the reduced induced closed subscheme structure. First of all, note that as $\dim(Z) < \dim(X)$, Z satisfies the conclusion of Theorem 4.7. We define $\hat{G} = i_*\hat{H} \oplus G$, where $\hat{H} \in \mathbf{D}^{\mathrm{b}}(\mathrm{coh}(Z))$ is the object coming from Z satisfying Theorem 4.7. As $F \in (\mathbf{D}_Z(\mathrm{Qcoh}\, X)^{\geq 0} \cap \mathbf{D}_Z(\mathrm{Qcoh}\, X)^{\leq t}) \star \overline{\langle \mathbb{R}j_*j^*\mathcal{O}_X \rangle}^{[-n,n]}$ and $\mathbb{R}j_*j^*\mathcal{O}_X \in \overline{\langle G \rangle}^{[-C,C]} \star (\mathbf{D}_Z(\mathrm{Qcoh}\, X)^{\geq -1} \cap \mathbf{D}_Z(\mathrm{Qcoh}\, X)^{\leq t})$ from the above paragraph, we would be done if we can show that for all $p \leq q$, there exists an integer $L_{p,q}$ such that,

$$\mathbf{D}_Z(\mathrm{Qcoh}\, X)^{\geq p} \cap \mathbf{D}_Z(\mathrm{Qcoh}\, X)^{\leq q} \subseteq \overline{\langle \hat{G} \rangle}^{[-L_{p,q}, L_{p,q}]}$$

But, as the conclusion of Theorem 4.7 holds for Z, we get this from Lemma 4.9, completing the proof. □

Corollary 4.11. *Let X be a Noetherian finite-dimensional scheme such that each integral closed subscheme is J-0. Then, there exists an object $\hat{G} \in \mathbf{D}^{\mathrm{b}}(\mathrm{coh}\, X)$ and a positive integer N such that $\mathbf{D}(\mathrm{Qcoh}\, X)^{\geq 0} = \overline{\langle \hat{G} \rangle}^{[-N,N]} \star \mathbf{D}(\mathrm{Qcoh}\, X)^{\geq 1}$, see Definition 2.1.*

Proof. Let $F \in \mathbf{D}(\mathrm{Qcoh}\, X)^{\geq 0}$. Then, we have the triangle $F^{\leq 0} \to F \to F^{\geq 1} \to \Sigma F^{\leq 0}$ we get from the standard t-structure on $\mathbf{D}(\mathrm{Qcoh}\, X)$. So, we have that $F^{\leq 0} \in \mathrm{Qcoh}\, X$ and $F^{\geq 1} \in \mathbf{D}(\mathrm{Qcoh}\, X)^{\geq 1}$. But, by Theorem 4.7, there exists $N \geq 0$ such that $\mathrm{Qcoh}\, X \subseteq \overline{\langle \hat{G} \rangle}^{[-N,N]}$. So, $F \in \overline{\langle \hat{G} \rangle}^{[-N,N]} \star \mathbf{D}(\mathrm{Qcoh}\, X)^{\geq 1}$. □

The following result is known, see for example [DL24, Theorem 1.1] and [ELS20, Theorem 4.15], but we give an alternate proof using Corollary 4.11.

Theorem 4.12. *Let X be a Noetherian finite-dimensional scheme such that each integral closed subscheme is J-0. Then, $\mathbf{K}(\mathrm{Inj}\, X)$ has a single compact generator. In fact, the object $\hat{G}$ of Corollary 4.11 is a compact generator of $\mathbf{K}(\mathrm{Inj}\, X)$.*

In particular, $\hat{G}$ is a classical generator for $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}\, X)$.

Proof. Immediate from Proposition 4.5(i) using Corollary 4.11. □

Theorem 4.13. *Let X be a Noetherian finite-dimensional scheme such that each integral closed subscheme is J-0. Then $\mathbf{K}(\mathrm{Inj}\, X)$ is a weakly co-approximable triangulated category, see Definition 2.13. Further, the standard co-t-structure lies in the preferred equivalence class, see Definition 2.9.*

Proof. Immediate from Proposition 4.5(ii), (iii) using Corollary 4.11. □

We also have a noncommutative version of the above results.

Theorem 4.14. *Let X be a Noetherian finite-dimensional J-2 scheme, and $\mathcal{A}$ any coherent $\mathcal{O}_X$-algebra. Then $\mathbf{K}(\mathrm{Inj}\, \mathcal{A})$ is a weakly co-approximable triangulated category, see Definition 2.13. Further, the standard co-t-structure lies in the preferred equivalence class, see Definition 2.9.*

Proof. Immediate from Proposition 4.5(ii), (iii) using Corollary A.6. □

We now prove the weak co-quasiapproximability for the mock homotopy category of projectives $\mathbf{K}_m(\mathrm{Proj}\, X)$. We will follow the notational conventions from [Mur08].

Proposition 4.15. *Let X be a Noetherian, separated, finite dimensional scheme such that every integral closed subscheme is J-0. Then, the mock homotopy category of projectives $\mathbf{K}_m(\mathrm{Proj}\, X)$ is weakly co-quasiapproximable (Definition 2.13).*

Proof. We will be working with the co-t-structure (U,V) compactly generated by the set $U_\lambda(\operatorname{coh} X)^\circ$, where $U_\lambda(-)^\circ : \mathbf{D}^{\mathrm{b}}(\operatorname{coh} X)^{\mathrm{op}} \xrightarrow{\sim} \mathbf{K}_m(\operatorname{Proj} X)^{\mathrm{c}}$, see [Mur08, Theorem 7.4]. By [MR25, Lemma 7.23], the closure of the compacts with respect to this co-t-structure is given by $\mathbf{K}_m(\operatorname{Proj}\text{-}X)^+_{\mathrm{c}} = U_\lambda(D^-(\operatorname{coh} X)^\circ$, see Definition 2.10 and Convention 2.12.

Note that $\mathbf{K}_m(\operatorname{Proj}\text{-}X)^+_{\mathrm{c}} \cap \mathsf{U} \subseteq U_\lambda(\mathbf{D}(\operatorname{coh} X)^{\leq 1})^\circ$ by [Bon22, Theorem 2.3.4] and [Nee21, Proposition 1.9]. But, for any $F \in \mathbf{D}(\operatorname{coh} X)^{\leq 1}$, we have the truncation triangle $F^{\leq -1} \to F \to F^{\geq 0} \to \Sigma F^{\leq -1}$ in which $F^{\leq -1} \in \mathbf{D}(\operatorname{coh} X)^{\leq -1}$ and $F^{\geq 0} \in \mathbf{D}(\operatorname{coh} X)^{\leq 1} \cap \mathbf{D}(\operatorname{coh} X)^{\geq 0} \subseteq \overline{\langle G\rangle}^{[-N,N]}$ for some object $G \in \mathbf{D}^{\mathrm{b}}(\operatorname{coh} X)$ and some integer $N \geq 0$ by Theorem 4.7.

So, applying $U_\lambda(-)^\circ$ to the above triangle gives us the required approximating triangle. As the rest of the axioms of weak co-quasiapproximability are easily verified, we are done with the proof. □

5. Examples from algebraic geometry

In this section, we apply the results in to categories coming from algebraic geometry. To apply the results, we first need to compute the closure of the compacts in these situations.

The closure of the compacts. In this section, we will be working with compactly generated triangulated categories with a single compact generator G. We will equip the category with one of the two metrics of Example 2.5. Throughout the section after Lemma 5.1, we will only be considering the closure of the compacts (Definition 2.10, Convention 2.12) with respect to a metric coming from a t-structure or a co-t-structure in the preferred quasiequivalence class, see Definition 2.9.

We begin with the cases where we work with the generating sequence $\mathcal{G}$ given by $\mathcal{G}[i,i] = \{\Sigma^i G\}$, see Definition 2.3. The following lemma helps us in computing the closure of the compacts.

Lemma 5.1. *Let T be a compactly generated triangulated category with a single compact generator G and a co-t-structure (U,V) on T such that $\operatorname{Hom}(\Sigma^i G, \mathsf{U}) = 0$ for $i \gg 0$. Then,*

$$\mathsf{T}^+_{\mathrm{c}} = \{\underrightarrow{\operatorname{Hocolim}}\, E_i : E_i \in \mathsf{T}^{\mathrm{c}}, \{E_i \to E_{i+1}\}_{i\geq 1} \text{ such that } \operatorname{Cone}(E_i \to E_{i+1}) \in \Sigma^{-i-1}\mathsf{U}\}$$

where the closure of the compacts is with respect to the metric $\{\Sigma^{-n}\mathsf{U}\}$, see Convention 2.12.

In particular, the closure of the compacts with respect to any other co-t-structure $(\mathsf{U}',\mathsf{V}')$ such that $\Sigma^{-N}\mathsf{U} \cap \mathsf{T}^{\mathrm{c}} \subseteq \mathsf{U}' \cap \mathsf{T}^{\mathrm{c}} \subseteq \Sigma^{N}\mathsf{U} \cap \mathsf{T}^{\mathrm{c}}$ for some $N \geq 0$ is the same.

Proof. This is immediate from Lemma 2.11. □

Proposition 5.2. *Let $\mathcal{C}$ be a locally Noetherian Grothendieck abelian category such that $\mathbf{K}(\operatorname{Inj}\mathcal{C})$ is weakly co-approximable and the standard co-t-structure lies in the preferred quasiequivalence class, see Definition 2.9. Recall that the full subcategory of compact objects $\mathbf{K}(\operatorname{Inj}\mathcal{C})^{\mathrm{c}} \cong \mathbf{D}^{\mathrm{b}}(\operatorname{noeth}(\mathcal{C}))$ by* [Kra05, Proposition 2.3]*. Then,*

- $\mathbf{K}(\operatorname{Inj}\mathcal{C})^+_{\mathrm{c}} = \mathbf{D}^+(\operatorname{noeth}(X))$
- $\mathbf{K}(\operatorname{Inj}\mathcal{C})^- = \bigcup_{n\in\mathbb{Z}} \mathbf{K}(\operatorname{Inj}\mathcal{C})^{\leq n}$
- $\mathbf{K}(\operatorname{Inj}\mathcal{C})^{\mathrm{b}}_{\mathrm{c}} = \mathbf{D}^{\mathrm{b}}(\operatorname{noeth}\mathcal{C}) \cap \mathbf{K}^{\mathrm{b}}(\operatorname{Inj}\mathcal{C})$

with notation as in Convention 2.12.

Proof. We can compute the closure of the compacts Lemma 5.1 with respect to the standard co-t-structure as it lies in the preferred equivalence class.

Let $F \in \mathbf{K}(\operatorname{Inj}\mathcal{C})^+_c$ (Convention 2.12). Then by definition, for each integer n there exists a triangle $E_n \to F \to D_n \to \Sigma E_n$ with $E_n \in \mathbf{D}^b(\operatorname{noeth}\mathcal{C})$ and $\mathbf{K}(\operatorname{Inj}\mathcal{C})^{\geq n}$. This immediately implies that F has bounded below and coherent cohomology, that is, it belongs to $\mathbf{D}^+(\operatorname{noeth}(\mathcal{C}))$. Conversely, for any $F \in \mathbf{D}^+(\operatorname{noeth}(\mathcal{C}))$, brutal truncation triangles easily give us that $F \in \mathbf{K}(\operatorname{Inj}\mathcal{C})^+_c$.

It trivially follows from the definitions that $\mathbf{K}(\operatorname{Inj}\mathcal{C})^- = \bigcup_{n\in\mathbb{Z}} \mathbf{K}(\operatorname{Inj}\mathcal{C})^{\leq n}$. Finally, from the above computations, it follows that $\mathbf{K}(\operatorname{Inj}\mathcal{C})^b_c = \mathbf{D}^b(\operatorname{noeth}\mathcal{C}) \cap \mathbf{K}^b(\operatorname{Inj}\mathcal{C})$. □

Corollary 5.3. *Let X be a Noetherian, finite dimension scheme such that every integral closed subscheme is J-0. Then, with notation as in Convention 2.12,*

- $\mathbf{K}(\operatorname{Inj} X)^+_c = \mathbf{D}^+(\operatorname{coh} X)$
- $\mathbf{K}(\operatorname{Inj} X)^- = \bigcup_{n\in\mathbb{Z}} \mathbf{K}(\operatorname{Inj} X)^{\leq n}$
- $\mathbf{K}(\operatorname{Inj} X)^b_c = \mathbf{D}^b(\operatorname{coh} X) \cap \mathbf{K}^b(\operatorname{Inj} X)$

Suppose that X is further J-2. Then, for any coherent $\mathcal{O}_X$-algebra $\mathcal{A}$, we have that,

- $\mathbf{K}(\operatorname{Inj}\mathcal{A})^+_c = \mathbf{D}^+(\operatorname{coh}\mathcal{A})$
- $\mathbf{K}(\operatorname{Inj}\mathcal{A})^- = \bigcup_{n\in\mathbb{Z}} \mathbf{K}(\operatorname{Inj}\mathcal{A})^{\leq n}$
- $\mathbf{K}(\operatorname{Inj}\mathcal{A})^b_c = \mathbf{D}^b(\operatorname{coh}\mathcal{A}) \cap \mathbf{K}^b(\operatorname{Inj}\mathcal{A})$

Proof. Immediate from Proposition 5.2 using Theorem 4.13 and Theorem 4.14. □

Corollary 5.4. *Let X be a Noetherian, separated, finite dimensional scheme such that every integral closed subscheme is J-0. Theen, with notation as in Convention 2.12,*

- $\mathbf{K}_m(\operatorname{Proj} X)^+_c = U_\lambda(\mathbf{D}^-(\operatorname{coh} X))^\circ$
- $\mathbf{K}_m(\operatorname{Proj} X)^b_c = U_\lambda(\mathbf{D}^{\mathrm{perf}}(X))^\circ$

Proof. By Lemma 2.15, the co-t-structure defined in the proof of Proposition 4.15 lies in the preferred quasiequivalence class, see Definition 2.9. Hence, the closure of the compacts can be computed using that co-t-structure. Then, the fact that $\mathbf{K}_m(\operatorname{Proj} X)^+_c = U_\lambda(\mathbf{D}^-(\operatorname{coh} X))^\circ$ follows from [MR25, Lemma 7.23].

Finally, it is clear that $\mathbf{K}_m(\operatorname{Proj} X)^b_c = U_\lambda(\mathsf{S})^\circ$, where S is the full subcategory of $\mathbf{D}^-(\operatorname{coh} X)$ defined by,

$$\mathsf{S} = \{F \in \mathbf{D}^-(\operatorname{coh} X) : \operatorname{Hom}_{\mathsf{T}}(F, \mathbf{D}(\operatorname{coh} X)^{\leq -n}) = 0 \text{ for all } n \gg 0\}$$

But for any $F \in \mathbf{D}^-(\operatorname{coh} X)$, there exist triangles $E_n \to F \to D_n \to \Sigma E_n$ by [Nee24, Theorem 3.3] and [Sta24, Tag 0FDA]. But, if F is further in S, then the map $F \to D_n$ vanishes for large n, which implies $F \in \mathbf{D}^{\mathrm{perf}}(X)$. The other inclusion is straightforward. □

Now, we come to those cases where we work with the generating sequence $\mathcal{G}$ given by $\mathcal{G}[i,i] = \{\Sigma^{-i}G\}$. We begin with the following result from [Nee24].

Proposition 5.5. [Nee24, page 281] *Let X be a quasicompact, quasiseparated scheme, and Z a closed subset with quasicompact complement. Then, the derived category of sheaves with quasicoherent cohomology supported on Z, denoted $\mathbf{D}_{\mathrm{Qcoh}}(X)$, has a single compact generator and the full subcategry of compact objects is given by $\mathbf{D}^{\mathrm{perf}}_Z(X)$, which* *is the derived category of perfect complexes supported on Z. The closure of compact is given by,*

- $\mathbf{D}_{\mathrm{Qcoh},Z}(X)^{-}_{\mathrm{c}} = \mathbf{D}^{p}_{\mathrm{Qcoh},Z}(X)$
- $\mathbf{D}_{\mathrm{Qcoh},Z}(X)^{\mathrm{b}}_{\mathrm{c}} = \mathbf{D}^{p,b}_{\mathrm{Qcoh},Z}(X)$

with notation as in Convention 2.12.

$\mathbf{D}^{p}_{\mathrm{Qcoh},Z}(X)$ here denotes the full subcategory of $\mathbf{D}_{\mathrm{Qcoh},Z}(X)$ consisting of the pseudocohrent complexes, while $\mathbf{D}^{p,b}_{\mathrm{Qcoh},Z}(X)$ denotes the full subcategory of pseudocoherent complexes with bounded cohomology. If X is Noetherian, then these categories agree with the derived category of sheaves with bounded below and coherent cohomology supported on Z, denoted $\mathbf{D}^{-}_{\mathrm{coh},Z}(X)$, and the bounded derived category of sheaves with coherent cohomology supported on Z, which is denoted by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh},Z}(X)$, respectively.

There is a noncommutative version of the above computation, which we state now.

Proposition 5.6. *Let X be a Noetherian scheme, and $\mathcal{A}$ a coherent $\mathcal{O}_X$-algebra. Then, $\mathbf{D}_{\mathrm{Qcoh}}(\mathcal{A})$, has a single compact generator and the full subcategory of compact objects is given by $\mathbf{D}^{\mathrm{perf}}(\mathcal{A})$, which is the derived category of perfect complexes. The closure of compact is given by,*

- $\mathbf{D}_{\mathrm{Qcoh},Z}(\mathcal{A})^{-}_{\mathrm{c}} = \mathbf{D}^{-}_{\mathrm{coh}}(\mathcal{A})$
- $\mathbf{D}_{\mathrm{Qcoh},Z}(\mathcal{A})^{\mathrm{b}}_{\mathrm{c}} = \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A})$.

with notation as in Convention 2.12.

Proof. This follows easily from [DLMR24, Proposition 4.2]. □

We now consider a stacky example, which follows easily from a couple of results of [HLLP25, DLMRP25].

Proposition 5.7. *Let $\mathcal{X}$ be a Noetherian algebraic stack which is concentrated, that is, the canonical morphism $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ is concentrated, see* [HR17, Definition 2.4]. *Further assume $\mathcal{X}$ satisfies one of the following two conditions,*

- *$\mathcal{X}$ has quasi-finite and separated diagonal.*
- *$\mathcal{X}$ is a DM stack of characteristic zero.*

Then, $\mathbf{D}_{\mathrm{Qcoh}}(\mathcal{X})$ has a single compact generator and and the closure of compact is given by,

- $\mathbf{D}_{\mathrm{Qcoh}}(\mathcal{X})^{-}_{\mathrm{c}} = \mathbf{D}^{-}_{\mathrm{coh}}(\mathcal{X})$.
- $\mathbf{D}_{\mathrm{Qcoh}}(\mathcal{X})^{\mathrm{b}}_{\mathrm{c}} = \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{X})$.

with notation as in Convention 2.12

Proof. By [DLMRP25, Proposition 5.10] the standard t-structure on $\mathbf{D}_{\mathrm{Qcoh}}(\mathcal{X})$ lies in the preferred equivalence class, see Definition 2.9. Then, the required result follows from [HLLP25, Theorem A] which proves that for any $F \in \mathbf{D}^{-}_{\mathrm{coh}}(\mathcal{X})$ and every integer n, there exists a triangle $E_n \to F \to D_n \to \Sigma E_n$ with $E_n \in \mathbf{D}^{\mathrm{perf}}(\mathcal{X})$ and $D_n \in \mathbf{D}_{\mathrm{Qcoh}}(\mathcal{X})^{\leq -n}$. □

Applications. We now state the applications of the results proved in §3 using the computation of the closure of the compacts in the previous subsection. In what follows, for any triangulated subcategory $\mathsf{A} \subseteq \mathsf{T}^c$ for a compactly generated triangulated category T, $\mathrm{Coprod}(\mathsf{A})$ would denote the localising subcategory of T generated by A.

We begin with the following result for quasicompact, quasiseparated schemes. The corresponding statement for Noetherian schemes is mentioned in the remark following it.

Theorem 5.8. *Let X be a quasicompact and quasiseparated scheme with a closed subset Z such that $X \setminus \mathsf{Z}$ is quasicompact. Let $\langle \mathsf{A}, \mathsf{B} \rangle$ be a semiorthogonal decomposition on $\mathbf{D}^{\mathrm{perf}}_Z(X)$, which is the derived category of perfect complexes with cohomology supported on Z. Then,*

$$\langle \mathrm{Coprod}(\mathsf{A}) \cap \mathbf{D}^p_{\mathrm{Qcoh},Z}(X), \mathrm{Coprod}(\mathsf{B}) \cap \mathbf{D}^p_{\mathrm{Qcoh},Z}(X) \rangle$$

is a semiorthogonal decomposition on $\mathbf{D}^p_{\mathrm{Qcoh},Z}(X)$, where $\mathbf{D}^p_{\mathrm{Qcoh},Z}(X)$ denotes the derived category of pseudocoherent complexes supported on Z.

If we further assume that B is an admissible subcategory of $\mathbf{D}^{\mathrm{perf}}_Z(X)$, then,

$$\langle \mathrm{Coprod}(\mathsf{A}) \cap \mathbf{D}^{p,b}_{\mathrm{Qcoh},Z}(X), \mathrm{Coprod}(\mathsf{B}) \cap \mathbf{D}^{p,b}_{\mathrm{Qcoh},Z}(X) \rangle$$

is a semiorthogonal decomposition on $\mathbf{D}^{p,b}_{\mathrm{Qcoh},Z}(X)$, where $\mathbf{D}^{p,b}_{\mathrm{Qcoh},Z}(X)$ denotes the derived category of pseudocoherent complexes with bounded cohomology supported on Z.

Proof. This is immediate from Corollary 3.6 using Proposition 5.5. ☐

We mention what the result says in the Noetherian case now, as the categories involved might be more familiar to the reader.

Remark 5.9. Let X be a Noetherian scheme with a closed subset Z. Let $\langle \mathsf{A}, \mathsf{B} \rangle$ be a semiorthogonal decomposition on $\mathbf{D}^{\mathrm{perf}}_Z(X)$, which is the derived category of perfect complexes with cohomology supported on Z. Then,

$$\langle \mathrm{Coprod}(\mathsf{A}) \cap \mathbf{D}^-_{\mathrm{coh},Z}(X), \mathrm{Coprod}(\mathsf{B}) \cap \mathbf{D}^-_{\mathrm{coh},Z}(X) \rangle$$

is a semiorthogonal decomposition on $\mathbf{D}^-_{\mathrm{coh},Z}(X)$, where $\mathbf{D}^-_{\mathrm{coh},Z}(X)$ denotes the derived category of sheaves with bounded above and coherent cohomology supported on Z.

If we further assume that B is an admissible subcategory of $\mathbf{D}^{\mathrm{perf}}_Z(X)$, then,

$$\langle \mathrm{Coprod}(\mathsf{A}) \cap \mathbf{D}^{\mathrm{b}}_{\mathrm{coh},Z}(X), \mathrm{Coprod}(\mathsf{B}) \cap \mathbf{D}^{\mathrm{b}}_{\mathrm{coh},Z}(X) \rangle$$

is a semiorthogonal decomposition on $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh},Z}(X)$, where $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh},Z}(X)$ denotes the derived category of sheaves with bounded and coherent cohomology supported on Z.

Theorem 5.10. *Let X be a Noetherian finite-dimensional scheme such that each integral closed subscheme is J-0. Let $\langle \mathsf{A}, \mathsf{B} \rangle$ be a semiorthogonal decomposition on $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}(X))$. Then,*

$$\langle \mathrm{Coprod}(\mathsf{A}) \cap \mathbf{D}^+(\mathrm{coh}\, X), \mathrm{Coprod}(\mathsf{B}) \cap \mathbf{D}^+(\mathrm{coh}\, X) \rangle$$

is a semiorthogonal decomposition on $\mathbf{D}^+(\mathrm{coh}\, X)$.

If we further assume that B is an admissible subcategory of $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}(X))$, then,

$$\langle \mathsf{A} \cap \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Inj}\, X), \mathsf{B} \cap \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Inj}\, X) \rangle$$

defines a semiorthogonal decomposition on $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Inj}\, X)$*, where* $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Inj}\, X) = \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \cap \mathbf{K}^{\mathrm{b}}(\mathrm{Inj}\, X)$ *denotes the full subcategory of complexes with finite injective dimension in the bounded derived category of sheaves with coherent cohomology.*

Proof. This is immediate from Corollary 3.7 using Proposition 5.2. □

Theorem 5.11. *Let* X *be a Noetherian finite-dimensional scheme such that each integral closed subscheme is J-0. Let* $\langle \mathsf{A}, \mathsf{B} \rangle$ *be a semiorthogonal decomposition on* $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}(X))$*. Then,*

$$\langle \mathrm{Coprod}(\mathsf{A}) \cap \mathbf{D}^{-}(\mathrm{coh}\, X), \mathrm{Coprod}(\mathsf{B}) \cap \mathbf{D}^{-}(\mathrm{coh}\, X) \rangle$$

is a semiorthogonal decomposition on $\mathbf{D}^{-}(\mathrm{coh}\, X)$*.*

If we further assume that B *is an admissible subcategory of* $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}(X))$*, then,*

$$\langle \mathsf{A} \cap \mathbf{D}^{\mathrm{perf}}(X), \mathsf{B} \cap \mathbf{D}^{\mathrm{perf}}(X) \rangle$$

is a semiorthogonal decomposition on $\mathbf{D}^{\mathrm{perf}}(X)$*.*

Proof. This is immediate from Corollary 3.7 using Corollary 5.4. □

We now state the noncommutative analogues of some of the above results.

Theorem 5.12. *Let* X *be a Noetherian scheme, and* $\mathcal{A}$ *a coherent* $\mathcal{O}_X$*-algebra. Let* $\langle \mathsf{A}, \mathsf{B} \rangle$ *be a semiorthogonal decomposition on* $\mathbf{D}^{\mathrm{perf}}(\mathcal{A})$*. Then,*

$$\langle \mathrm{Coprod}(\mathsf{A}) \cap \mathbf{D}^{-}_{\mathrm{coh}}(\mathcal{A}), \mathrm{Coprod}(\mathsf{B}) \cap \mathbf{D}^{-}_{\mathrm{coh}}(\mathcal{A}) \rangle$$

is a semiorthogonal decomposition on $\mathbf{D}^{-}_{\mathrm{coh}}(\mathcal{A})$*.*

If we further assume that B *is an admissible subcategory of* $\mathbf{D}^{\mathrm{perf}}(\mathcal{A})$*, then,*

$$\langle \mathrm{Coprod}(\mathsf{A}) \cap \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}), \mathrm{Coprod}(\mathsf{B}) \cap \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}) \rangle$$

is a semiorthogonal decomposition on $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A})$*.*

Proof. Immediate from Corollary 3.6 using Proposition 5.6. □

Theorem 5.13. *Let* X *be a Noetherian finite-dimensional J-2 scheme, and* $\mathcal{A}$ *a coherent* $\mathcal{O}_X$*-algebra. Let* $\langle \mathsf{A}, \mathsf{B} \rangle$ *be a semiorthogonal decomposition on* $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}(\mathcal{A}))$*. Then,*

$$\langle \mathrm{Coprod}(\mathsf{A}) \cap \mathbf{D}^{+}(\mathrm{coh}\, \mathcal{A}), \mathrm{Coprod}(\mathsf{B}) \cap \mathbf{D}^{+}(\mathrm{coh}\, \mathcal{A}) \rangle$$

is a semiorthogonal decomposition on $\mathbf{D}^{+}(\mathrm{coh}\, \mathcal{A})$*.*

If we further assume that B *is an admissible subcategory of* $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}(\mathcal{A}))$*, then,*

$$\langle \mathrm{Coprod}(\mathsf{A}) \cap \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Inj}\, \mathcal{A}), \mathrm{Coprod}(\mathsf{B}) \cap \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Inj}\, \mathcal{A}) \rangle$$

is a semiorthogonal decomposition on $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Inj}\, \mathcal{A})$*, where* $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Inj}\, \mathcal{A}) = \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}) \cap \mathbf{K}^{\mathrm{b}}(\mathrm{Inj}\, \mathcal{A})$ *denotes the full subcategory of complexes with finite injective dimension in the bounded derived category of sheaves with coherent cohomology.*

Proof. This is immediate from Corollary 3.7 using Proposition 5.2. □

Finally, we get the following result for stacks.

Theorem 5.14. *Let* $\mathcal{X}$ *be a Noetherian algebraic stack which is concentrated, that is, the canonical morphism* $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ *is concentrated, see* [HR17, Definition 2.4]*. Further assume* $\mathcal{X}$ *satisfies one of the following two conditions,*

- $\mathcal{X}$ *has quasi-finite and separated diagonal.*

- *$\mathcal{X}$ is a DM stack of characteristic zero.*

Let $\langle \mathsf{A}, \mathsf{B}\rangle$ be a semiorthogonal decomposition on $\mathbf{D}^{\text{perf}}(\mathcal{X})$. Then,

$$\langle \text{Coprod}(\mathsf{A}) \cap \mathbf{D}^-_{\text{coh}}(\mathcal{X}), \text{Coprod}(\mathsf{B}) \cap \mathbf{D}^-_{\text{coh}}(\mathcal{X})\rangle$$

is a semiorthogonal decomposition on $\mathbf{D}^-_{\text{coh}}(\mathcal{X})$.

If we further assume that B is an admissible subcategory of $\mathbf{D}^{\text{perf}}(\mathcal{X})$, then,

$$\langle \text{Coprod}(\mathsf{A}) \cap \mathbf{D}^b_{\text{coh}}(\mathcal{X}), \text{Coprod}(\mathsf{B}) \cap \mathbf{D}^b_{\text{coh}}(\mathcal{X})\rangle$$

is a semiorthogonal decomposition on $\mathbf{D}^b_{\text{coh}}(\mathcal{X})$.

Proof. This is immediate from Corollary 3.6 using Proposition 5.7. □

6. A new bijection result

In this section, we will prove Theorem A. For that, we begin by proving the following result, which is proven by a combination of the results of §3 along with the representability theorems of [MR25].

Theorem 6.1. *Let T be a $\mathcal{G}$-quasiapproximable triangulated category ([MR25, Definition 4.2]) for a finite generating sequence $\mathcal{G}$ such that $\check{\mathcal{G}}$ is compressed (Definitions 2.3, 2.4 and 3.2). Further, assume that T is R-linear for a commutative Noetherian ring R, and that $\text{Hom}_{\mathsf{T}}(G, G')$ is a finitely generated R-module for all G, G' in $\bigcup_{n\in\mathbb{Z}} \mathcal{G}[n,n]$. Then, given a right admissible subcategory of the compacts, we get an admissible subcategory of the closure of the compacts and a left admissible subcategory of the bounded objects in the closure of the compacts given as below,*

$$\begin{aligned} \text{RAdm}(\mathsf{T}^c) &\longrightarrow \text{Adm}(\overline{\mathsf{T}^c}) \\ \mathsf{A} &\longmapsto \mathsf{A}^{\perp} \cap \overline{\mathsf{T}^c} \end{aligned} \qquad\qquad \begin{aligned} \text{RAdm}(\mathsf{T}^c) &\longrightarrow \text{LAdm}(\mathsf{T}^b_c) \\ \mathsf{A} &\longmapsto \mathsf{A}^{\perp} \cap \mathsf{T}^b_c \end{aligned}$$

Furthermore, the assignment $\text{RAdm}(\mathsf{T}^c) \longrightarrow \text{Adm}(\overline{\mathsf{T}^c})$ defined above is injective.

Proof. Let $\langle \mathsf{B}, \mathsf{A}\rangle$ be a semiorthogonal decomposition on T^c. Then, by Theorem 3.5 $\langle \text{Coprod}(\mathsf{B}) \cap \overline{\mathsf{T}^c}, \text{Coprod}(\mathsf{A}) \cap \overline{\mathsf{T}^c}\rangle$ is a semiorthogonal decomposition on $\overline{\mathsf{T}^c}$. Note that $\text{Coprod}(\mathsf{B}) \cap \overline{\mathsf{T}^c} = \mathsf{A}^{\perp} \cap \overline{\mathsf{T}^c}$. This shows that $\mathsf{A}^{\perp} \cap \overline{\mathsf{T}^c}$ is left admissible, and so it remains to show it is right admissible.

We define the Serre subcategory $\mathcal{S} \subseteq \prod_{i\in\mathbb{Z}} \text{Mod}(R)$ to be the collection of all sequences of finitely generated modules which vanish for large enough integers. That is, a sequence of modules $\{M_i\}$ lies in $\mathcal{S}$ if and only if M_i is finitely generated for all i, and $M_i = 0$ for $i \gg 0$. Note that $\mathcal{S}$ is compressed, see Definition 3.2. Furthermore, $\overline{\mathsf{T}^c} = \mathsf{T}_{\mathcal{S},\check{\mathcal{G}}}$ where $\mathsf{T}_{\mathcal{S},\check{\mathcal{G}}}$ is as defined in Definition 3.9. That $\overline{\mathsf{T}^c} \subseteq \mathsf{T}_{\mathcal{S},\check{\mathcal{G}}}$ is immediate from [MR25, Lemma 6.9]. Conversely, let $F \in \mathsf{T}_{\mathcal{S},\check{\mathcal{G}}}$. By [MR25, Theorem 6.22], there exists $E \in \overline{\mathsf{T}^c}$ such that $\mathcal{Y}(E)|_{\mathsf{T}^c} \cong \mathcal{Y}(F)|_{\mathsf{T}^c}$. As $E \in \overline{\mathsf{T}^c}$, by [MR25, Lemma 6.2(3)] and [Nee25b, Lemma 6.8], there exists a map $f : E \to F$ realizing this isomorphism. But, as T is compactly generated, this implies that f is an isomorphism giving us that $F \in \overline{\mathsf{T}^c}$, and hence we have shown that $\overline{\mathsf{T}^c} = \mathsf{T}_{\mathcal{S},\check{\mathcal{G}}}$. Therefore, $\mathsf{A}^{\perp} \cap \overline{\mathsf{T}^c}$ is a right admissible subcategory of $\overline{\mathsf{T}^c}$ by Theorem 3.10, which is what we needed to show. We get the second map by instead working with the Serre subcategory $\mathcal{S}'$ consisting of all sequences $\{M_i\}$ of finitely generated R-modules such that $M_i = 0$ for $|i| \gg 0$.

Finally, the injectivity of the assignment is immediate from the observation that for any right admissible subcategory A of T^c, we have that ${}^{\perp}(\mathsf{A}^{\perp} \cap \overline{\mathsf{T}^c}) \cap \mathsf{T}^c = \text{Coprod}(\mathsf{A}) \cap \overline{\mathsf{T}^c} \cap \mathsf{T}^c = \text{smd}(\text{coprod}(\mathsf{A})) = \mathsf{A}$ where the first equality follows from the first paragraph of the proof, the second from [Nee21, Proposition 1.9], and the third from the fact that A is a thick subcategory. □

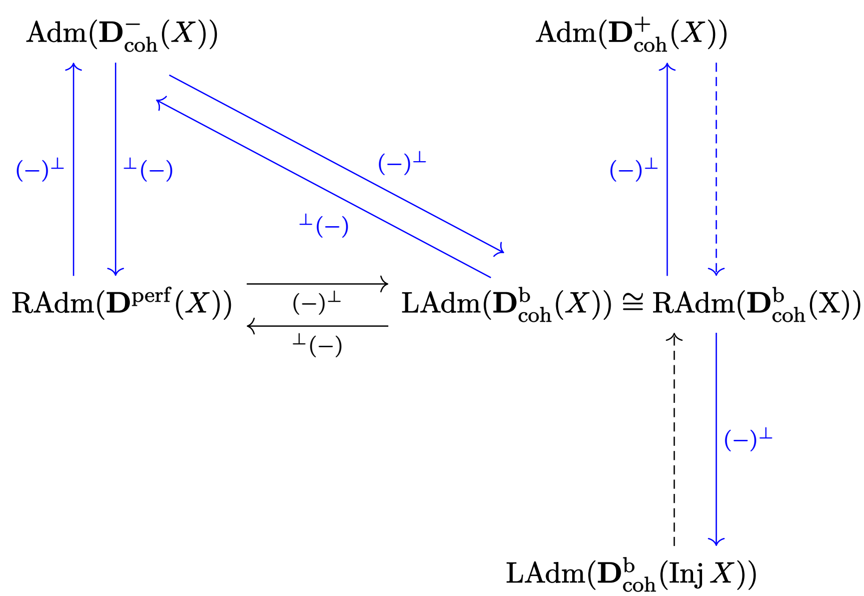
We note here that the conditions of Theorem 6.1 are satisfied for a large class of naturally occuring triangulated categories

Remark 6.2. Let R be a Noetherian ring, and T be an R-linear triangulated category with a single compact generator G such that $\text{Hom}(G, \Sigma^i G)$ is a finitely generated R-module for all integers i. Then, T satisfies the conditions of Theorem 6.1 if T is quasiapproximable or co-quasiapproximable. In particular, they hold for,

(1) The derived category of quasicoherent sheaves of any scheme which is proper over a Noetherian ring.
(2) The derived category of quasicoherent sheaves of any coherent $\mathcal{O}_X$-algebra where X is any scheme which is proper over a Noetherian ring.
(3) The derived category of modules over any proper connective DG-algebra.
(4) Any recollement of triangulated categories as in (1)-(3).
(5) The homotopy category of injectives of any quasiexcellent scheme which is proper over a finite-dimensional Noetherian ring.
(6) The homotopy category of injectives of any coherent $\mathcal{O}_X$-algebra where X is any quasiexcellent scheme which is proper over a finite-dimensional Noetherian ring.
(7) The mock homotopy category of projectives for any quasiexcellent scheme which is proper over a finite-dimensional Noetherian ring.

We now prove Theorem A.

Theorem 6.3. *Let X be a quasiexcellent scheme which is proper over a Noetherian finite-dimensional ring R. Then, the solid maps in the diagram below exist,*



where $(-)^{\perp}$ and ${}^{\perp}(-)$ by abuse of notation denotes the full subcategory of the appropriate category consisting of objects which are left right orthogonal to the given (left and/or right) admissible category respectively.

If we assume that X has a dualizing complex, then the dashed arrows exist too. In either of these cases, whenever arrows exist in both the directions between a pair in the diagram, they give a bijection between the corresponding sets.

Finally, we note here that the arrows in black were known before, and appear in [Bon24, Theorem 1.3.II.2], *while the arrows in blue are new.*

Proof. Let X be a quasiexcellent scheme which is proper over a Noetherian finite-dimensional ring R. We prove the solid morphisms exist. Throughout the proof, for any full subcategory D of $\mathbf{D}_{\mathrm{Qcoh}}(X)$ and any integer n, $\mathsf{D}^{\leq n}$ and $\mathsf{D}^{\geq n}$ will denote the full subcategories $\mathsf{D} \cap \mathbf{D}_{\mathrm{Qcoh}}(X)^{\leq n}$ and $\mathsf{D} \cap \mathbf{D}_{\mathrm{Qcoh}}(X)^{\geq n}$ respectively. Here $\mathbf{D}_{\mathrm{Qcoh}}(X)^{\leq n}$ (resp. $\mathbf{D}_{\mathrm{Qcoh}}(X)^{\geq n}$) denotes the full subcategories of $\mathbf{D}_{\mathrm{Qcoh}}(X)$ consisting of complexes F with $H^i(F) = 0$ for $i > n$ (resp. $i < n$).

$$(1)\quad \mathrm{RAdm}(\mathbf{D}^{\mathrm{perf}}(X)) \underset{{}^{\perp}\mathsf{A}\cap\mathbf{D}^{\mathrm{perf}}(X) \leftarrow\!\shortmid \mathsf{A}}{\overset{\mathsf{A}\mapsto \mathsf{A}^{\perp}\cap \mathbf{D}^{-}_{\mathrm{coh}}(X)}{\rightleftarrows}} \mathrm{Adm}(\mathbf{D}^{-}_{\mathrm{coh}}(X))$$

Let H be a classical generator for $\mathbf{D}^{\mathrm{perf}}(X)$. We define the generating sequence $\mathcal{G}$ by setting $\mathcal{G}[i,i] = \{\Sigma^{-i}H\}$, whose inverse is a compressed generating sequence by Theorem 3.3. By [MR25, Theorem 4.14, Proposition 5.3] and [Nee25b, Example 4.6], $\mathbf{D}_{\mathrm{Qcoh}}(X)$ is $\mathcal{G}$-quasiapproximable. Therefore, for any right admissible subcategory A of $\mathbf{D}^{\mathrm{perf}}(X)$, $\mathsf{A}^{\perp} \cap \mathbf{D}^{-}_{\mathrm{coh}}(X)$ is an admissible subcategory of $\mathbf{D}^{-}_{\mathrm{coh}}(X)$ by Theorem 6.1, which gives us the first map.

Conversely, let A be an admissible subcategory of $\mathbf{D}^{-}_{\mathrm{coh}}(X)$. This gives us a recollement as follows,

$$\mathsf{A} \;\underset{i^!}{\overset{i^*}{\leftleftarrows}}\!\xrightarrow{\;i_*\;} \mathbf{D}^{-}_{\mathrm{coh}}(X) \;\underset{j_*}{\overset{j_!}{\leftleftarrows}}\!\xrightarrow{\;j^*\;} \mathsf{B}$$

It is enough to show that $i_*i^*E, j_!j^*E \in \mathbf{D}^{\mathrm{perf}}(X)$ for all $E \in \mathbf{D}^{\mathrm{perf}}(X)$. We begin by recalling the following well-known description of $\mathbf{D}^{\mathrm{perf}}(X)$,

$$\mathbf{D}^{\mathrm{perf}}(X) = \{F \in \mathbf{D}^{-}_{\mathrm{coh}}(X) : \mathrm{Hom}(F, \mathbf{D}^{-}_{\mathrm{coh}}(X)^{\leq -n}) = 0 \text{ for all } n \gg 0\}$$

see for example [Nee25a, proof of Proposition 5.9(2)]. Now, it is again well-known that any $F \in \mathbf{D}^{-}_{\mathrm{coh}}(X)^{\leq n}$ can be written as a homotopy limit of a sequence $F_1 \to F_2 \to \cdots$ of objects in $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(X)^{\leq n}$. This can be shown easily using the canonical truncation and [BN93, Remark 2.3]. By [Aok21, Main Theorem], there exists a strong generator G for $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, which in turn implies that there exists $N > 0$ such that

$$\overline{\langle G\rangle}^{(-\infty, n-N]} \cap \mathbf{D}^{-}_{\mathrm{coh}}(X) \subseteq \mathbf{D}^{-}_{\mathrm{coh}}(X)^{\leq n} \subseteq \overline{\langle G\rangle}^{(-\infty, n+N]} \cap \mathbf{D}^{-}_{\mathrm{coh}}(X)$$

for all integers n by [Nee25b, Lemma 9.4]. Finally, we can choose an integer B such that $i_*i^!G, j_*j^*G \in \mathbf{D}^{-}_{\mathrm{coh}}(X)^{\leq B}$. And so, for any $E \in \mathbf{D}^{\mathrm{perf}}(X)$ and $F \in \mathbf{D}^{-}_{\mathrm{coh}}(X)^{\leq n}$, we have that

$$\mathrm{Hom}(i_*i^*E, F) = \mathrm{Hom}(E, i_*i^!F) = \mathrm{Hom}(E, i_*i^! \underrightarrow{\mathrm{Holim}}\, F_i)$$

for a sequence $\{F_i\}$ in $\mathbf{D}^{\mathrm{b}}_{\mathsf{coh}}(X)^{\leq n}$. But, note that $F_i \in \mathbf{D}^{\mathrm{b}}_{\mathsf{coh}}(X)^{\leq n} \subseteq \langle G \rangle^{(-\infty, n+N]}$, and hence $i_* i^* F_i \in \mathbf{D}^{-}_{\mathsf{coh}}(X)^{\leq n+N+B}$. And so, $\mathrm{Hom}(E, \mathbf{D}^{-}_{\mathsf{coh}}(X)^{\leq n+N+B}) = 0$ for $n \ll 0$ as $E \in \mathbf{D}^{\mathrm{perf}}(X)$, and therefore $\mathrm{Hom}(i_* i^* E, \mathbf{D}^{-}_{\mathsf{coh}}(X)^{\leq n}) = 0$ for $n \ll 0$, which in turn implies that $i_* i^* E \in \mathbf{D}^{\mathrm{perf}}(X)$. We can similarly show that $j_! j^* E \in \mathbf{D}^{\mathrm{perf}}(X)$.

This gives us the two maps we were after, and it remains to show that these in fact do give a bijection. For this, we begin by observing that for any right admissible subcategory A of $\mathbf{D}^{\mathrm{perf}}(X)$, we have that ${}^{\perp}(\mathsf{A}^{\perp} \cap \mathbf{D}^{-}_{\mathsf{coh}}(X)) = \mathrm{Coprod}(\mathsf{A}) \cap \mathbf{D}^{-}_{\mathsf{coh}}(X)$ by Remark 5.9. So,

$$ {}^{\perp}(\mathsf{A}^{\perp} \cap \mathbf{D}^{-}_{\mathsf{coh}}(X)) \cap \mathbf{D}^{\mathrm{perf}}(X) = \mathrm{Coprod}(\mathsf{A}) \cap \mathbf{D}^{-}_{\mathsf{coh}}(X) \cap \mathbf{D}^{\mathrm{perf}}(X) = \mathsf{A} $$

where the final equality holds as $\mathrm{Coprod}(\mathsf{A}) \cap \mathbf{D}^{\mathrm{perf}}(X) = \mathrm{smd}(\mathsf{A}) = \mathsf{A}$ by [Nee21, Proposition 1.9]. Conversely, let $\widetilde{\mathsf{B}}$ be an admissible subcategory of $\mathbf{D}^{-}_{\mathsf{coh}}(X)$. Let $\widetilde{\mathsf{A}} = {}^{\perp}\widetilde{\mathsf{B}} \cap \mathbf{D}^{-}_{\mathsf{coh}}(X)$. The paragraph above shows that $\langle \mathsf{B}, \mathsf{A} \rangle$ is a semiorthogonal decomposition on $\mathbf{D}^{\mathrm{perf}}(X))$ where $\mathsf{A} := \widetilde{\mathsf{A}} \cap \mathbf{D}^{\mathrm{perf}}(X))$ and $\mathsf{B} := \widetilde{\mathsf{B}} \cap \mathbf{D}^{\mathrm{perf}}(X))$. And so,

$$ ({}^{\perp}\widetilde{\mathsf{B}} \cap \mathbf{D}^{\mathrm{perf}}(X))^{\perp} \cap \mathbf{D}^{-}_{\mathsf{coh}}(X) = \mathsf{A}^{\perp} \cap \mathbf{D}^{-}_{\mathsf{coh}}(X) = \mathrm{Coprod}(\mathsf{B}) \cap \mathbf{D}^{-}_{\mathsf{coh}}(X) = \widetilde{\mathsf{B}} $$

which proves the required bijection.

$$ (2) \quad \mathrm{LAdm}(\mathbf{D}^{\mathrm{b}}_{\mathsf{coh}}(X)) \; \underset{\mathsf{A}^{\perp} \cap \mathbf{D}^{\mathrm{b}}_{\mathsf{coh}}(X) \longleftarrow\!\shortmid \mathsf{A}}{\overset{\mathsf{A} \mapsto {}^{\perp}\mathsf{A} \cap \mathbf{D}^{-}_{\mathsf{coh}}(X)}{\rightleftarrows}} \; \mathrm{Adm}(\mathbf{D}^{-}_{\mathsf{coh}}(X)) $$

Let G be a strong generator for $\mathbf{D}^{\mathrm{b}}_{\mathsf{coh}}(X)$, which exists by [Aok21, Main Theorem]. We define the generating sequence $\mathcal{G}$ given by $\mathcal{G}[i,i] = \{\Sigma^i G\}$ for all integers i. This is a compressed generating sequence by Theorem 3.3. Furthermore, the mock homotopy category of projectives is $\mathcal{G}$-quasiapproximable by [MR25, Theorem 7.25 and Theorem 7.10]. Finally, the compacts and the closure of the compacts are given isomorphic to $\mathbf{D}^{\mathrm{b}}_{\mathsf{coh}}(X))^{\mathrm{op}}$ and $\mathbf{D}^{-}_{\mathsf{coh}}(X))^{\mathrm{op}}$, see Corollary 5.4. And so, given a left admissible subcategory A of $\mathbf{D}^{\mathrm{b}}_{\mathsf{coh}}(X))$, we have that ${}^{\perp}\mathsf{A} \cap \mathbf{D}^{-}_{\mathsf{coh}}(X)$ is an admissible subcategory of $\mathbf{D}^{-}_{\mathsf{coh}}(X)$ by Theorem 6.1.

Conversely, let A be an admissible subcategory of $\mathbf{D}^{-}_{\mathsf{coh}}(X)$. This gives us a recollement as follows,

$$ \mathsf{A} \;\underset{i^!}{\overset{i^*}{\leftleftarrows}}\!\xrightarrow{i_*}\; \mathbf{D}^{-}_{\mathsf{coh}}(X) \;\underset{j_*}{\overset{j_!}{\leftleftarrows}}\!\xrightarrow{j^*}\; \mathsf{B} $$

It is enough to show that $i^! i_* E, j_* j^* E \in \mathbf{D}^{\mathrm{b}}_{\mathsf{coh}}(X))$ for all $E \in \mathbf{D}^{\mathrm{b}}_{\mathsf{coh}}(X))$. Observe that,

$$ \mathbf{D}^{\mathrm{b}}_{\mathsf{coh}}(X)) = \{F \in \mathbf{D}^{-}_{\mathsf{coh}}(X)) : \mathrm{Hom}(\mathbf{D}^{-}_{\mathsf{coh}}(X))^{\leq -n}, F) = 0 \text{ for all } n \gg 0\} $$

Let $H \in \mathbf{D}^{\mathrm{perf}}(X))$ be a classical generator. Then, there exist positive integers N and B such that,

- $i_* i^* H, j_! j^* H \in \mathbf{D}^{-}_{\mathsf{coh}}(X))^{\leq B}$ as $\mathbf{D}^{-}_{\mathsf{coh}}(X)) = \bigcup_{n \geq 1} \mathbf{D}^{-}_{\mathsf{coh}}(X))^{\leq n}$,
- for any object $F \in \mathbf{D}^{-}_{\mathsf{coh}}(X))^{\leq n}$, there exists a sequence $\{F_i\}$ of objects in $\langle H \rangle^{(-\infty, n+A]}$ mapping to F such that $\underrightarrow{\mathrm{Hocolim}}\, F_i \cong F$ by [Nee25b, Corollary 2.14].

Now, let $E \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(X))$ and $F \in \mathbf{D}^{-}_{\mathrm{coh}}(X))^{\leq n}$. As noted above, there exists a sequence $\{F_i\}$ in $\langle H\rangle^{(-\infty,n+A]}$ mapping to F such that $\underrightarrow{\mathrm{Hocolim}}\, F_i \cong F$, hence,

$$\mathrm{Hom}(F, i^! i_* E) = \mathrm{Hom}(i_* i^* F, E) = \mathrm{Hom}(i_* i^* \underrightarrow{\mathrm{Hocolim}}\, F_i, E)$$

But, as $i_* i^* F_i \in \mathbf{D}^{-}_{\mathrm{coh}}(X))^{\leq n+B+N}$, we get that $\mathrm{Hom}(F, i^! i_* E) = 0$ for $n \ll 0$, which gives us that $i^! i_* E \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(X))$. Similarly, $j_* j^* E \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(X))$, which is what we needed to show. Finally, the bijection follows similar to (1).

(3) $\mathrm{RAdm}(\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(X)) \xrightarrow{\mathsf{A} \longmapsto \mathsf{A}^{\perp} \cap \mathbf{D}^{+}_{\mathrm{coh}}(X)} \mathrm{Adm}(\mathbf{D}^{+}_{\mathrm{coh}}(X))$

Let G be a strong generator for $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, which exists by [Aok21, Main Theorem]. We again define the generating sequence $\mathcal{G}$ given by $\mathcal{G}[i,i] = \{\Sigma^i G\}$ for all integers i which is a compressed generating sequence by Theorem 3.3. The homotopy category of injectives is $\mathcal{G}$-quasiapproximable by [MR25, Theorem 7.19, Theorem 7.10, and Proposition 5.3].

(4) $\mathrm{RAdm}(\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(X)) \xrightarrow{\mathsf{A} \longmapsto \mathsf{A}^{\perp} \cap \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Inj}\, X)} \mathrm{LAdm}(\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Inj}\, X))$

Follows exactly the same as (3), using the other part of Theorem 6.1.

Finally, the dashed arrows exist in the presence of a dualizing complex as in that case, we have that $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Inj}\, X) \cong \mathbf{D}^{\mathrm{perf}}(X))$ and $\mathbf{D}^{+}_{\mathrm{coh}}(X) \cong \mathbf{D}^{-}_{\mathrm{coh}}(X)$. ☐

Appendix A. Co-approximability for noncommutative coherent algebras

In this appendix, we will prove some technical results, which are used to prove the co-approximability of the homotopy category of injectives for a coherent algebra over a scheme, see Theorem 4.14. We begin with a definition.

Definition A.1. A **Noether algebra** is a pair $(X, \mathcal{A})$ where X is a Noetherian scheme and $\mathcal{A}$ is a coherent $\mathcal{O}_X$-algebra. We will denote the structure map by $\pi : \mathcal{O}_X \to \mathcal{A}$. The abelian category of quasicoherent (resp. coherent) $\mathcal{A}$-modules is denoted by $\mathrm{Qcoh}\,\mathcal{A}$ (resp. $\mathrm{coh}\,\mathcal{A}$). The full subcategory of injective quasicoherent $\mathcal{A}$-modules is denoted by $\mathrm{Inj}\,\mathcal{A}$.

The co-approximability result will follow from Proposition 4.5 using the following theorem. We prove this theorem by a sequence of lemma, analogous to the proof of Theorem 4.7.

Theorem A.2. *Let $(X, \mathcal{A})$ be a Noether algebra for a finite dimensional J-2 scheme X. Let T be any triangulated subcategory of $\mathbf{D}(\mathrm{Qcoh}\,\mathcal{A})$ which is closed under the canonical truncations. Then, there is an object $\hat{G}$ in $\mathbf{D}^{\mathrm{b}}(\mathrm{coh}\,\mathcal{A})$ such that,*

1. *There exists an integer $N \geq 0$ such that $\mathsf{T} \cap \mathrm{Qcoh}\,\mathcal{A} \subseteq \overline{\langle \hat{G} \rangle}^{[-N,N]}$, see Definition 2.1.*
2. *There exist integers $N_{p,q} \geq 0$ for each pair of integers $p \leq q$ such that*

$$\mathsf{T} \cap \mathbf{D}(\mathrm{Qcoh}\,\mathcal{A})^{\geq p} \cap \mathbf{D}(\mathrm{Qcoh}\,\mathcal{A})^{\leq q} \subseteq \overline{\langle \hat{G} \rangle}^{[-N_{p,q}, N_{p,q}]}$$

see Definition 2.1.

Remark A.3. As with Theorem 4.7, it is clear that it suffices to show Theorem A.2 for $\mathsf{T} = \mathbf{D}(\operatorname{Qcoh} \mathcal{A})$. Further, note that Theorem A.2(1) $\implies$ (2). This can be shown by an induction on $(q-p)$ analogous to Remark 4.8.

Lemma A.4. *Let $(X, \mathcal{A})$ be Noether algebra. If $i : Z \to X$ is a closed immersion such that the conclusion of Theorem A.2 holds for $(Z, i^*(\mathcal{A}))$, then the conclusion of Theorem A.2 holds for $(X, \mathcal{A})$ with $\mathsf{T} = \mathbf{D}_Z(\operatorname{Qcoh} \mathcal{A})$.*

Proof. Same as proof of Lemma 4.9. □

Lemma A.5. *Let $(X, \mathcal{A})$ be a Noether algebra, and let $i_1 : Z_1 \to X$ and $i_2 : Z_2 \to X$ be closed subschemes such that $X = Z_1 \cup Z_2$ topologically. If the conclusion of Theorem A.2 holds for $(Z_1, i_1^*(\mathcal{A}))$ and $(Z_2, i_2^*(\mathcal{A}))$, then it holds for $(X, \mathcal{A})$.*

Proof. Same as proof of Lemma 4.10. □

Now we can prove Theorem A.2.

Proof of Theorem A.2. Let $(X, \mathcal{A})$ be a Noether algebra, where X is a Noetherian finite-dimensional J-2 scheme . We start by observing some reductions we can do. First of all, by Remark A.3, it is enough to prove Theorem A.2(1) for $\mathsf{T} = \mathbf{D}(\operatorname{Qcoh} \mathcal{A})$. Further, we can assume X is irreducible using Lemma A.5, and that X is reduced by Lemma A.4. So we can assume that the scheme X is reduced and irreducible, that is, the scheme is integral.

We now proceed by induction on the dimension of the scheme X. For the base case, let X be a integral scheme such that $\dim(X) = 0$. Then X is just $\operatorname{Spec}(k)$ for a field k and $\mathcal{A} = \widetilde{A}$ for a finite-dimensional k-algebra A. Then, by Lemma A.4, we can replace A by $A/J(A)$, where $J(A)$ is the Jacobson radical. As this algebra is semisimple, the result holds.

Now, we assume the result is known for dimension smaller than $\dim(X)$. By [ELS20, Proposition 4.10], there exists a non-empty affine open set $U = \operatorname{Spec}(R)$, a two sided ideal $I \subseteq \mathcal{A}(U)$ and $\mathcal{O}_X(U)$ algebras $A_1, \cdots, A_r$ for some $r \geq 1$, such that each A_i is an Azumaya algebra over its center $Z(A_i)$, each center $Z(A_i)$ is a regular ring, and there is an isomorphism $\mathcal{A}(U)/I \cong A_1 \times \cdots \times A_r$. Let the corresponding open immersion be $j : U \to X$. Let $F \in \operatorname{Qcoh} \mathcal{A}$. Then, the restriction to the open set U, $j^*F \in \operatorname{Qcoh}(j^*(\mathcal{A})) \cong \operatorname{Mod}(\mathcal{A}(U))$. Now, as I is a nilpotent ideal, there is some $n \geq 1$ such that $I^n = 0$. This gives us a filtration $0 = I^n M \subseteq I^{n-1} M \subseteq \cdots \subseteq IM \subseteq M$. The corresponding triangles allow us to replace the algebra $\mathcal{A}(U)$ by the algebra $A = \mathcal{A}(U)/I$.

Now, $\operatorname{Mod}(A) = \operatorname{Mod}(A_1) \times \cdots \times \operatorname{Mod}(A_r)$. As $Z(A_i)$ is a regular ring of finite Krull dimension, let $N = \max\{\dim(Z(A_i)) : 1 \leq i \leq r\}$. Then, the global dimension of each A_i is less than or equal to N by [AG60, Theorems 1.8 and 2.1]. So, j^*F must have a projective resolution of length less than or equal to N. This gives us that $j^*F \in \overline{\langle A \rangle}^{[-n,n]} = \overline{\langle j^* \mathcal{A} \rangle}^{[-n,n]}$.

Consider the triangle $F' \to F \to \mathbb{R}j_* j^* F \to \Sigma F'$, where the second map is the unit of adjunction. Note that when restricted to U, we get that the unit of adjunction is an isomorphism, and so $j^*(F') \cong 0$. Let $Z = X - U$. Then, $F' \in \mathbf{D}_Z(\operatorname{Qcoh} \mathcal{A})$. By [Lip09, Proposition 3.9.2], there exists $t \geq 0$ such that $\mathbb{R}j_* j^* F \in \mathbf{D}(\operatorname{Qcoh} \mathcal{A})^{\geq 0} \cap \mathbf{D}(\operatorname{Qcoh} \mathcal{A})^{\leq t-1}$. So, as $F' \in (\Sigma^{-1} \mathbb{R}j_* j^* F) \star F$ we get that,

$$F' \in \mathbf{D}_Z(\operatorname{Qcoh} \mathcal{A})^{\geq 0} \cap \mathbf{D}_Z(\operatorname{Qcoh} \mathcal{A})^{\leq t}$$

This gives us that,

$$F \in \left(\mathbf{D}_Z(\operatorname{Qcoh}\mathcal{A})^{\geq 0} \cap \mathbf{D}_Z(\operatorname{Qcoh}\mathcal{A})^{\leq t}\right) \star \overline{\langle \mathbb{R}j_*j^*\mathcal{A}\rangle}^{[-n,n]}$$

as $F \in F' \star \mathbb{R}j_*j^*F$, and as $\mathbb{R}j_*j^*F \in \mathbb{R}j_*(\overline{\langle j^*\mathcal{A}\rangle}^{[-n,n]}) \subseteq \overline{\langle \mathbb{R}j_*j^*\mathcal{A}\rangle}^{[-n,n]}$ by the previous paragraph. Note that the integer t can be chosen independent of the choice of F in $\operatorname{Qcoh}\mathcal{A}$.

Consider the triangle $Q \to \mathcal{A} \to \mathbb{R}j_*j^*\mathcal{A} \to \Sigma Q$ in $\mathbf{D}(\operatorname{Qcoh}\mathcal{A})$ coming from the unit of adjunction. Note that when restricted to U, we get that the unit of adjunction is an isomorphism, and so $j^*(Q) \cong 0$. Therefore, $\Sigma Q \in \mathbf{D}_Z(\operatorname{Qcoh}\mathcal{A})^{\geq -1} \cap \mathbf{D}_Z(\operatorname{Qcoh}\mathcal{A})^{\leq t}$. Further, $\mathcal{A} \in \langle G\rangle^{[-C,C]}$ for a compact generator $G \in \mathbf{D}^{\mathrm{perf}}(\mathcal{A})$ and some positive integer C. So, $\mathbb{R}j_*j^*\mathcal{A} \in \langle G\rangle^{[-C,C]} \star (\mathbf{D}_Z(\operatorname{Qcoh}\mathcal{A})^{\geq -1} \cap \mathbf{D}_Z(\operatorname{Qcoh}\mathcal{A})^{\leq t})$.

Now, let $i : Z \to X$ be the closed immersion where Z is given the reduced induced closed subscheme structure. First of all, note that as $\dim(Z) < \dim(X)$, $(Z, i^*(\mathcal{A}))$ satisfies the conclusion of Theorem A.2. We define $\hat{G} = i_*\hat{H} \oplus G$, where $\hat{H} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(i^*(\mathcal{A}))$ is the object coming from $(Z, i^*(\mathcal{A}))$ satisfying Theorem A.2. As $F \in (\mathbf{D}_Z(\operatorname{Qcoh}\mathcal{A})^{\geq 0} \cap \mathbf{D}_Z(\operatorname{Qcoh}\mathcal{A})^{\leq t}) \star \overline{\langle \mathbb{R}j_*j^*\mathcal{A}\rangle}^{[-n,n]}$ and $\mathbb{R}j_*j^*\mathcal{A} \in \overline{\langle \mathcal{G}\rangle}^{[-C,C]} \star (\mathbf{D}_Z(\operatorname{Qcoh}\mathcal{A})^{\geq -1} \cap \mathbf{D}_Z(\operatorname{Qcoh}\mathcal{A})^{\leq t})$ from the above paragraph, we would be done if we can show that for all $p \leq q$, there exists an integer $L_{p,q}$ such that,

$$\mathbf{D}_Z(\operatorname{Qcoh}\mathcal{A})^{\geq p} \cap \mathbf{D}_Z(\operatorname{Qcoh}\mathcal{A})^{\leq q} \subseteq \overline{\langle \hat{G}\rangle}^{[-L_{p,q}, L_{p,q}]}$$

But, as the conclusion of Theorem A.2 holds for $(Z, i^*(\mathcal{A}))$, we get this from Lemma A.4, completing the proof. □

Corollary A.6. *Let $(X, \mathcal{A})$ be a Noether algebra with X a finite-dimensional J-2 scheme. Then, there exists an object $\hat{G} \in \mathbf{D}^{\mathrm{b}}(\operatorname{coh}\mathcal{A})$ and a positive integer N such that $\mathbf{D}(\operatorname{Qcoh}\mathcal{A})^{\geq 0} = \overline{\langle \hat{G}\rangle}^{[-N,N]} \star \mathbf{D}(\operatorname{Qcoh}\mathcal{A})^{\geq 1}$, see Definition 2.1.*

Proof. Let $F \in \mathbf{D}(\operatorname{Qcoh}\mathcal{A})^{\geq 0}$. Then, we have the triangle $F^{\leq 0} \to F \to F^{\geq 1} \to \Sigma F^{\leq 0}$ we get from the standard t-structure on $\mathbf{D}(\operatorname{Qcoh}\mathcal{A})$. So, we have that $F^{\leq 0} \in \operatorname{Qcoh}\mathcal{A}$ and $F^{\geq 1} \in \mathbf{D}(\operatorname{Qcoh}\mathcal{A})^{\geq 1}$. But, by Theorem A.2, there exists $N \geq 0$ such that $\operatorname{Qcoh}\mathcal{A} \subseteq \overline{\langle \hat{G}\rangle}^{[-N,N]}$. So, $F \in \overline{\langle \hat{G}\rangle}^{[-N,N]} \star \mathbf{D}(\operatorname{Qcoh}\mathcal{A})^{\geq 1}$, which is what we needed to show. □

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